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COLLEGE ALGEBRA

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PREFACE

In this book, which is based in part on our *Algebra for College Students*, we have followed the suggestions of a number of our friends in condensing the earlier portions of the work in order to facilitate a more rapid review of the topics from elementary algebra. The time which is thus gained may, at the instructor's discretion, be spent on the advanced topics of the last several chapters.

We have attempted to keep as far as is practicable those features of the *Algebra for College Students* which have been found to be good. The explanations have been kept simple and deliberate but full enough to meet the needs of the students. We trust that we have also met the instructor's requirements for rigor.

Despite the fact that there is considerable emphasis on the techniques of algebra, efficiency in manipulation alone is not the aim of the book. The major stress is on the important underlying ideas.

A number of departures from traditional treatments will be noted. Chapter 10 contains more geometrical material than is found in the usual algebra text; for example, the concept of similar triangles is explained and used to prove the Pythagorean theorem. The practical significance of variation as met in the laboratory is employed to impress the student with the importance of this topic.

Chapter XI consists of a discussion of certain aspects of computation with approximate numbers, a subject of vital concern to students of science and technology. This topic is either omitted or given scant treatment in most algebra books.

Because five-place tables of common logarithms are customarily used for practical calculations, the computational work of Chapter 12 makes use of such a table (found in the appendix). In dealing with the topic of change of base for logarithms, we have tried to avoid formulas that the student is likely to forget after the first quiz. Instead, we have emphasized the fundamental fact that corresponding logarithms to different bases are proportional to each other. A short explanation of the use of tables of natural logarithms has been included.

In Chapter 15, we have inserted a brief exposition of vectors and have made use of the vector representation in order to give the student a more concrete feeling for complex numbers. The idea of "turning and stretching" serves to give a visual picture of the operation of multiplication of complex numbers. These ideas are introduced without the use of the trigonometric form, which is brought in later for those students who have the requisite preparation.

As additional material beyond that covered in the *Algebra for College Students*, we have included the classical solutions of the cubic and the quartic equation, the trigonometric solution of the cubic in the case of three real roots, a chapter on partial fractions, and one on finite differences.

The chapter on partial fractions will be found to be quite different from that in the average college algebra. The idea of splitting off one partial fraction at a time has been made the basis for the theory of partial fractions. The same idea has also served as a vehicle to introduce not only the usual device of undetermined coefficients but also certain practical short cuts which are quite successful in the classroom as well as in actual applications.

A noteworthy feature of the book consists of the 120 sets of carefully graded exercises, ranging from relatively easy drill problems to problems that test the abilities of the best students.

It is a pleasure to acknowledge again the extremely valuable editorial aid furnished us by Rinehart and Company and their editor, Doctor C. V. Newsom. The results of this editing are reflected throughout the book. We are grateful for the many excellent suggestions made by the mathematics staff of the University of Michigan; in particular, by Professors E. D. Rainville and P. S. Jones. We are especially indebted to our colleagues at the University of Colorado for their many important contributions to this book.

J. R. B.

L. C. S.

Boulder, Colorado
January 1953

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Chapter 1

REVIEW OF FUNDAMENTAL IDEAS

1. Introduction

Algebra is a continuation and a generalization of arithmetic. We shall therefore consider briefly certain fundamental ideas from arithmetic, and shall restate these ideas in algebraic language.

We assume that the student is familiar with the mechanics of *addition, subtraction, multiplication,* and *division* of numbers. The symbols $+$ (plus), $-$ (minus), $\times$ (times), $\div$ (divided by), and $=$ (equals) are used in algebra with the meanings that are customary in arithmetic.

Perhaps the most important characteristic of algebra is its use of symbols other than the usual numerals to stand for numbers; the symbols commonly employed are letters of the alphabet. Such numbers are called **general,** or **literal, numbers,** and they may be assigned specific values, or they may represent numbers whose values, at first unknown, are to be found.

2. The Fundamental Laws of Arithmetic

The operations with numbers obey certain general rules which are called the **Fundamental Laws of Arithmetic.** In elementary mathematics, these laws are accepted as justified by experience. In more advanced work, we find that we can introduce and use systems of "numbers" which do not obey all the rules of arithmetic. Hence, the laws of arithmetic are taken as fundamental *postulates* or *assumptions* that characterize the numbers and the operations of ordinary algebra. For the present the student may consider the following statements to

be concerned with the positive numbers only, that is, with the integers and fractions of elementary arithmetic.

(1) **The Commutative Law of Addition:** *The sum of a given set of numbers is a definite fixed number, regardless of the order in which the separate numbers are added.* In symbols, we have

$$a + b = b + a.$$

EXAMPLE. $5 + 7 = 7 + 5 = 12.$

(2) **The Associative Law of Addition:** *In adding a given set of numbers, the separate numbers may be grouped in any manner whatsoever.* In symbols, we write

$$(a + b) + c = a + (b + c).$$

EXAMPLE. $(4 + 3) + 5 = 4 + (3 + 5).$

(3) **The Commutative Law of Multiplication:** *The product obtained by multiplying two numbers together is independent of the order in which the numbers are taken.* Thus,

$$ab = ba.$$

EXAMPLE. $4 \times 3 = 3 \times 4 = 12.$

(4) **The Associative Law of Multiplication:** *In multiplying three (or more) numbers together, the numbers may be arbitrarily grouped.* In algebraic language this postulate is written

$$(ab)c = a(bc).$$

EXAMPLE. $(3 \times 4) \times 5 = 3 \times (4 \times 5).$

(5) **The Distributive Law of Multiplication with Respect to Addition:** *If the sum of several numbers is to be multiplied by another number, the product of each of the several numbers and the multiplier may be obtained and these products added.* Thus

$$a(b + c) = ab + ac.$$

EXAMPLE. $3 \times (4 + 8) = (3 \times 4) + (3 \times 8).$

We shall have frequent occasion in the succeeding pages to refer to these five laws.

3. Subtraction, Zero, and the Negative Numbers

Instead of setting up a separate set of rules for subtraction, it is simpler to think of this operation as the *inverse* of addition. This means

that the question

$$13 - 7 = ?$$

is to be regarded as the equivalent of

$$7 + ? = 13.$$

More generally, we define the difference of two positive numbers a and b, in the stated order, as the number x that satisfies the relation

$$a = b + x,$$

and we write

$$x = a - b.$$

If a is greater than b, then x is one of our familiar positive numbers. If a is less than b, or if a is equal to b, then there is no positive number x that represents the difference $a - b$. In arithmetic, we drop the matter here; but, in algebra, we are not satisfied with such a state of affairs. We use this situation as a jumping-off place to define some new numbers.

(1) In the case where $a = b$, we define the difference $a - b$ to be the number *zero*, that is, for any positive number a,

$$a - a = 0.$$

One of the difficulties many students have is caused by their failure to regard *zero* as a *number*—that number, in fact, which is obtained by subtracting any number from itself. The importance of this remark will appear as we proceed.

It follows from the definition (1) that if the number zero is added to or subtracted from any other number a, the result is exactly a:

$$a + 0 = a,$$

and

$$a - 0 = a.$$

(2) In the case where a is less than b, we define the difference $a - b$ to be a *negative* number and agree to write it in the form

$$-(b - a).$$

For example,

$$5 - 8 = -(8 - 5) \quad = -3,$$

and

$$\frac{1}{4} - \frac{1}{2} = -\left(\frac{1}{2} - \frac{1}{4}\right) = -\frac{1}{4}.$$

Accordingly, exactly one negative number will correspond to each of the ordinary (positive) numbers of arithmetic. Furthermore, each negative

number can be represented in many ways in the form $-(a - b)$, where a and b are positive numbers and a is greater than b.

We shall define the operations with negative numbers by insisting that the five fundamental laws of arithmetic remain valid. We first examine the meaning of the addition of a negative number as in the expression $7 + (-2)$. By definition, we may write

$$-2 = -(6 - 4) = 4 - 6,$$

so that

$$7 + (-2) = 7 + 4 - 6 = 5.$$

Thus the result of adding (-2) is the same as that of subtracting $(+2)$, that is,

$$+(-2) = -(+2).$$

Since the same reasoning can be applied to the addition of any negative number, we agree that

$$+(-b) = -(+b).$$

It is customary in algebra to understand the plus sign before a number when no other sign is written. Consequently, the preceding equation is usually written

$$(-b) = -(b).$$

This equation says that *the addition of any negative number is equivalent to the subtraction of the corresponding positive number.*

We now have from the commutative law that

$$a - b = (-b) + a,$$

or, in words, *the subtraction of the positive number* b *is equivalent to the addition of the negative number* $-$b. It follows at once that

$$-(a - b) = +(-a + b) = -a + b,$$

or verbally, *parentheses preceded by a minus sign may be removed if the signs of all terms within the parentheses are reversed.*

Let us consider the product $(a)(-b)$ where a and b are both positive. We have, by the commutative law of multiplication, that

$$(a)(-b) = (-b)(a).$$

Furthermore, $(-b) = -(b)$, so that

$$(a)(-b) = (-b)(a) = -(b)(a) = -(ba) = -ab,$$

and

$$(-a)(-b) = -(a)(-b) = -(-ab) = ab.$$

Thus, from the fundamental laws follows the *rule of signs in multiplication: The product of two nonzero numbers is positive if the numbers have like signs, negative if the numbers have unlike signs.*

This rule can easily be extended to the product of more than two factors so that we may state: *The product of several nonzero factors is positive if none or an even number of factors is negative; the product is negative if an odd number of factors is negative.*

Another immediate consequence of the fundamental laws is the important rule: *The result of multiplying any number by zero is zero.* This fact may be seen from the following demonstration:

$$\begin{aligned} a \times 0 &= a(b - b) \\ &= ab - ab \\ &= 0. \end{aligned}$$

The student should make clear to himself that this answer follows even if $a = 0$. It is important to notice that a product cannot be zero unless one of its factors is zero.

4. Division and More about Zero

It is convenient, as in the case of subtraction, to think of division as an inverse process, the inverse of multiplication. Thus, the division

$$a \div b = q,$$

or, what is the same thing,

$$\frac{a}{b} = q,$$

is to be considered as a restatement of the multiplication

$$a = bq.$$

The quotient of two numbers is a third number, *if there is one,* which, multiplied by the divisor, gives the dividend.

First, we can see that if there is any quotient, it is generally unique, that is, there is not more than one number q such that $a = bq$. Thus, *if* b *is not zero,* and x and y are two numbers such that $bx = a$ and $by = a$, then

$$bx - by = a - a = 0,$$

or

$$b(x - y) = 0.$$

But b is not zero; hence

$$x - y = 0,$$

and

$$x = y.$$

Next, we consider the exceptional case $b = 0$. With $b = 0$, we know that $bq = 0$ for every number q. Consequently, *if* a *is not zero,* then bq cannot be equal to a for any number q, and the indicated division $a \div 0$ is impossible! If, on the other hand, $a = 0$, then our equation reads $0 = 0 \times q$, which is true for every number q! Thus, if the divisor is zero, either the division is impossible or else no unique quotient is defined. We are forced by common sense to make the rule: *The use of zero as a divisor is forbidden.*

If a and b are positive integers, and a is some positive integer, say m, times b, then the symbol a/b is simply the integer m. If there is no such integer, then a/b defines a common fraction. The numbers which we have thus far considered consist of the positive and negative integers and common fractions and the number zero. These numbers are called the **rational** numbers because it is possible to write each one as the ratio a/b of two integers (including zero as a special case). *With the exception of division by zero,* the four fundamental operations, addition, subtraction, multiplication, and division can be performed on any two rational numbers and the result is always a rational number. For this reason, the set of rational numbers is said to be **closed** with respect to the fundamental operations.

EXERCISES 1

1. Add 2589 and 3742; then add 4761 and 9238. Now add the two sums. Check by adding the four numbers all at once. What law is illustrated?

2. Explain how Figure 1 may be used to illustrate the commutative law of addition.

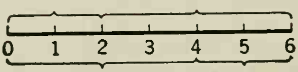


Fig. 1

3. A bank teller, in finding the value of a package of bills, first separates them into piles, each of one denomination. He then counts each pile separately and adds the results. What law of arithmetic does he use in each step?

4. Find the value of $6.5 \times 7.8 \times 1.6$. Check your answer by doing the problem in two ways. What law or laws have you used?

5. What laws of arithmetic can you illustrate by means of the diagram in Figure 2 below?

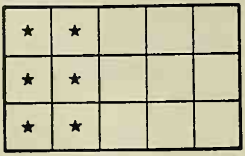


Fig. 2

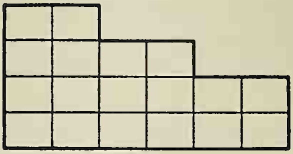


Fig. 3

6. Show that the diagram in Figure 3 can be used to illustrate the distributive law.

7. In order to multiply a number by 25, we may move the decimal point two places to the right and divide the resulting number by 4. Can you justify this procedure on the basis of the fundamental laws? Use the idea of this example to give a rule for dividing by 25.

8. In order to divide a number by 42, we may divide by 6, and then divide the resulting quotient by 7. Justify this procedure on the basis of the fundamental laws.

9. Give several ways in which the division of a number by 36 may be broken down into simpler divisions.

10. Is there any operation which is the inverse of multiplication by zero? Explain.

11. Give a definition of the number zero. What is the result of multiplying zero by itself? Answer the same question for division.

12. In a certain problem, several numbers are all to be divided by the same divisor, and the results are to be added. Is there a better way to obtain the final result? What law justifies the second method?

13. What are the five fundamental laws of arithmetic?

14. Explain why $\frac{10}{12}$ has the same value as $\frac{5}{6}$.

15. Name the operation which is the inverse of (*a*) multiplication; (*b*) subtraction; (*c*) division; (*d*) addition.

5. The Number Scale

Consider an ordinary Fahrenheit thermometer which bears a scale giving readings *above* zero and *below* zero. It is convenient to indicate a reading of 65° *above zero* as $+65°$ and a reading of 10° *below zero* as $-10°$. Notice carefully that the plus and minus signs are no longer signs of operation but indicate explicitly the *sense*, or *direction*, in which the numbers are to be taken. Another important point is that the two symbols indicate *opposite* directions. This means that if, contrary to the usual practice, we should choose to have $-65°$ stand for 65° above zero, then $+10°$ must indicate 10° below zero.

Following the line of thought suggested by this example, we make a diagram as follows: Draw a horizontal straight line (considered as endless in both directions), and fix upon it some point to serve as an origin, or zero point (see Figure 4). Choose a unit length, and select one of the directions on the line to be considered positive. This direction is usually

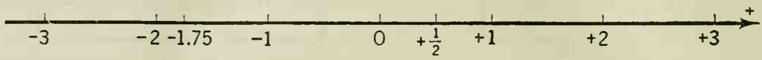


Fig. 4

taken to the right; to the left is then negative. Lay off the unit length from the zero point in the positive direction, and mark the point so obtained 1. Repeat, starting from the point marked 1, to obtain the point to be marked 2, and again for the point to be marked 3, and so forth in an obvious manner. Now, starting from the zero point, lay off the unit length in the negative direction to find the point to be marked -1, and repeat as before to obtain the points to be labeled -2, -3, and so forth.

Clearly, the point midway between 0 and 1 represents the number $\frac{1}{2}$; the point three fourths of the way from -1 to -2 represents the number -1.75. Any number whose point is to the right of the zero point is a **positive number**; any number whose point is to the left of the zero point is a **negative number.**

Thus, we have a line on which, corresponding to any signed number, there is exactly one point. It also appears that to each point on the line there corresponds exactly one signed number. The importance of this correspondence lies in the fact that the line furnishes us with an exact picture of the number system of elementary algebra. This line, together with the numbers associated with its points, is a **number scale;** the operations that we perform with numbers may be pictured as operations performed upon the number scale.

6. The Order Properties of the Signed Numbers

An inspection of the number scale shows that the positive numbers are arranged in order of increasing magnitude from left to right. For example, the number 20 has its point farther to the right than does the number 5; this corresponds to the fact that 20 is greater than 5. Accordingly, we agree to extend this "ordering" to the entire number system. That this idea is in agreement with common experience may be seen from the fact that $-10°$F. is a higher temperature than $-60°$F.

Any number whose point on the number scale lies farther to the right than that of a second number is said to be greater than the second number. The second number is also said to be less than the first number.

The symbols of order used in this connection are

$>$ for *is greater than;*

and $<$ for *is less than.*

The student should observe that in both cases the sign points toward the smaller number.

Illustrations:

$$20 > 5 \quad \text{or} \quad 5 < 20;$$
$$1 > -10 \quad \text{or} \quad -10 < 1;$$
$$0 > -100 \quad \text{or} \quad -100 < 0;$$
$$-10 > -12 \quad \text{or} \quad -12 < -10.$$

The symbol $\neq$ is used to mean *is not equal to.* This sign is less restrictive than either of the symbols $>$ and $<$.

EXERCISES 2

Replace the symbol $\neq$ in each of the following statements by the proper one of the symbols $>$ or $<$:

1. $2 \neq 7$ **2.** $-9 \neq -13$ **3.** $-4 \neq 15$ **4.** $-12 \neq -6$

5. $-3 \neq -5$ **6.** $9 \neq 13$ **7.** $0.444 \neq \frac{4}{9}$ **8.** $0.666 \neq \frac{2}{3}$

Carry out the indicated operations in each of the following exercises. In computations involving the fundamental operations, it is commonly agreed that multiplications and divisions take precedence over addition and subtraction unless grouping symbols indicate otherwise.

9. $(9 - 13) - (-7 + 12) + (17 - 11) - (-9 - 10)$

10. $(14 - 19) + (-8 - 3) - (-10 - 19) - (27 - 21)$

11. $19 - \{9 - [2 - (12 - 27)]\}$

12. $7 - \{-4 - [-3 - (6 - 17)]\}$

13. $(11 - 15) \times (-2) - \{22 - [7 - (15 - 17)]\}$

14. $(8 - 14) \times (-3) - \{-13 - [12 - (24 - 19)]\}$

15. $[(37 - 19) - (35 - 14)] \times [(15 - 11) - (12 - 17)]$

16. $[(18 - 24) - (42 - 51)] \times [(23 - 19) - (71 - 69)]$

17. $\dfrac{(-4) \times (-5) + 8}{-11 + (-3) \times (-6)} \times \dfrac{(-4) \times (-6) \times (-5)}{(-3) \times (-40)}$

18. $\dfrac{7 \times (-6) + 78}{(-4) \times (-3) - (-6)} \times \dfrac{10 + 5 \times (-3)}{4 \times (-6) + 25}$

7. Algebraic Expressions and Their Evaluation

For the present, any group of symbols made up of literal and specific numbers properly connected by signs of operation will be called an **algebraic expression.** We shall frequently use only the word "expression" to mean the same thing.

The *value* of an algebraic expression is the number it represents when the letters involved are given specific numerical values. The process of finding this value is known as **evaluating** the expression.

EXAMPLE 1. Evaluate the expression $\dfrac{u+v}{u-v}$ if $u = 5$ and $v = -3$.

Solution: For the given values, we have

$$\frac{u+v}{u-v} = \frac{5+(-3)}{5-(-3)} = \frac{2}{8} = \frac{1}{4}. \quad \textit{Ans.}$$

Observe that the assigned number values replace each letter without affecting the preceding sign of operation.

One of the most important skills needed for proficiency in algebra is the ability to translate verbal statements into the corresponding algebraic symbols. Before the student is able to translate complete statements, he must be able to write algebraic expressions to represent various quantities. With this in mind, the following examples should be studied carefully.

EXAMPLE 2. A man walked at the rate of 4 mph for m hr and at the rate of 3 mph for the next n hr. Set up the expression for the total number of miles the man walked.

Solution: If m were 2, the man would have walked 4×2 or 8 miles at the rate of 4 mph. In the same way, if n were 5, he would have walked 3×5 or 15 miles at the rate of 3 mph. Under these assumptions, the total number of miles would be

$$(4 \times 2) + (3 \times 5) = 23.$$

Without using specific values for m and n, we see that in m hr, at the rate of 4 mph, the man walked $4m$ miles. Similarly, in n hr, at the rate of 3 mph, he walked $3n$ miles. Hence, if d is the total number of miles, we must have

$$d = 4m + 3n. \quad \textit{Ans.}$$

EXAMPLE 3. At a certain entertainment, a tickets were sold at 60 cents each, b tickets at 40 cents, and c tickets at 30 cents. Express in dollars the total amount collected.

Solution: Let N represent the number of dollars collected. Then, $0.60a$ is the number of dollars obtained for the 60-cent tickets; $0.40b$ the corresponding number for the 40-cent tickets; and $0.30c$ for the 30-cent tickets. Hence, the total number of dollars is

$$N = 0.60a + 0.40b + 0.30c. \quad \textit{Ans.}$$

EXERCISES 3

Evaluate each of the following expressions for the values $a = -2$, $b = 3$, $c = -1$, $d = 4$, $x = -3$, and $y = 5$:

1. $(5y - 3x)(2y + 3x)$

2. $(4a + 2b)(c - 3d)$

3. $(2a - b - 2)(a + 3b + 1)$

4. $(a - b)(c - d)(x + y)$

5. $acx(ab - 2cd - xy)$

6. $4bcd(xy - 4ab + 4cd)$

7. $\dfrac{5cy - 2bx}{3ax + cd}$

8. $\dfrac{2ab + 3cd}{xy + 2bd}$

9. $\dfrac{(3x + y)(y - x)}{(x + y)(3x + 2y)}$

10. $\dfrac{(6c + 2d)(d - 4c)}{(8c + 3d)(d - 2c)}$

In each of the following, find the value of the quantity required in part (a). Then, by analyzing what you have done, write the algebraic expression for the quantity called for in the remaining parts of the problem. Do not fail to evaluate these expressions for the numbers given in part (a) as a check.

11. (*a*) The cost c in cents of eight folders at 7 cents each.
(*b*) The cost c in cents of eight folders at n cents each.
(*c*) The cost c in cents of k folders at n cents each.

12. (*a*) The area A of a rectangle with sides 6 in. and 7 in. long, respectively.
(*b*) The area A of a rectangle with sides a in. and b in. long, respectively.

13. (*a*) The distance d traveled by a motorist who drove for 5 hr at the rate of 55 mph.
(*b*) The distance d traveled by a motorist who drove for 5 hr at the rate of v mph.
(*c*) The distance d traveled by a motorist who drove for t hr at the rate of v mph.

14. (*a*) The speed v of an airplane which flies 750 miles in 2 hr.
(*b*) The speed v of an airplane which flies 750 miles in t hr.
(*c*) The speed v of an airplane which flies s miles in t hr.

15. (*a*) The volume V of a rectangular solid which is 9 in. long, 3 in. high, and 5 in. wide.

(*b*) The volume V of a rectangular solid which is a in. long, b in. high, and c in. wide.

16. (*a*) The value A in cents of five dimes and four nickels.
(*b*) The value A in cents of x dimes and y nickels.

17. (*a*) The value A in dollars of seven quarters, six half dollars, and five dollars.
(*b*) The value A in dollars of x quarters, y half dollars, and z dollars.

18. (*a*) The average price p of three articles which cost 14 cents, 19 cents, and 27 cents, respectively.
(*b*) The average price p of three articles which cost a cents, b cents, and c cents, respectively.

19. (*a*) The average price p of twelve articles: three at 12 cents each, four at 14 cents each, and five at 8 cents each.
(*b*) The average price p of $x + y + z$ articles: x at a cents each, y at b cents each, and z at c cents each.

20. (*a*) The fractional part f of a job a man can do in 5 days, if it takes him 9 days to do the entire job.
(*b*) The fractional part f of a job a man can do in a days if it takes him x days to do the entire job.

21. (*a*) The fractional part f of a job A and B can do in 4 days if A can do the job alone in 16 days and B can do it alone in 20 days.
(*b*) The fractional part f of a job A and B can do in t days if A can do the job alone in x days and B can do it alone in y days.

22. (*a*) The number of miles s two trains will be apart if they start at the same place and time and travel for 7 hr in opposite directions at speeds of 70 mph and 85 mph, respectively.
(*b*) The number of miles s two trains will be apart if they start at the same place and time and travel for h hr in opposite directions at speeds of x mph and y mph, respectively.

23. (*a*) The number of ounces w of pure silver in an alloy which weighs 16 oz and is 40 per cent silver.
(*b*) The number of ounces w of pure silver in an alloy which weighs x oz and is y per cent silver.

8. Addition and Subtraction of Simple Expressions

In an algebraic expression consisting of an indicated sum or difference of literal numbers, any number separated from the rest by plus signs is called a **term** of the expression.

In the expression $3a + b - 5c$, the terms are $3a$, b, and $-5c$. Notice that the last term is thought of as $+(-5c)$.

When a term is a product of a specific and a literal number, the specific number is called the **numerical coefficient** of the term. This coefficient will always be understood to include the sign immediately

preceding it. Again, in the preceding expression, the three terms have the numerical coefficients $+3$, $+1$, and -5, respectively.

In any algebraic expressions, terms whose *literal* factors are *identical* are said to be **like** terms. Thus, in the expressions $3a + 2b$ and $5a - 6b$, $3a$ and $5a$ are like terms; $2b$ and $-6b$ are like terms. If necessary, we shall speak of terms having different literal parts as **unlike** terms; $3a$ and $2b$ are unlike terms.

In order to add two or more algebraic expressions, we add the numerical coefficients of the like terms. The sums so obtained are the coefficients of the corresponding like terms in the result.

In practice, if several expressions are to be added, they are best written in columns with like terms in the same column (just as in an ordinary arithmetic addition where separate columns are reserved for units, tens, hundreds, and so forth).

EXAMPLE 1. Add the expressions $3a - 2b + 7c$, $9a + 15b - 25c$, and $-37a - 13b + 8c$. Check for $a = 4$, $b = 3$, and $c = 2$.

Solution: *Check:*

$$\begin{array}{rrr} 3a & -\ 2b & +\ 7c \\ 9a & +15b & -25c \\ -37a & -13b & +\ 8c \\ \hline -25a & +0\cdot b & -10c \end{array} \qquad\qquad \begin{array}{r} 20 \\ 31 \\ -171 \\ \hline -120 \end{array}$$

The answer is $-25a - 10c$. The term $0\cdot b$ is, of course, zero and does not have to be written in the result. The student should notice the check obtained by giving the letters specific values and evaluating the expressions. The sum of the numbers in the check column should be the same as the evaluated answer. This evaluation does not constitute a certain check. If the two sums do not agree, a mistake has been made; agreement of the sums, however, means only that the work is probably correct.

We have already observed that the subtraction of signed numbers may be effected by changing the sign of the number to be subtracted and adding. This remark is all that is needed to handle the subtraction of algebraic expressions.

EXAMPLE 2. From the sum of $2a + 7b - 15c$ and $5a - 8b + 3c$ subtract the sum of $9a + 3b - 10c$ and $-4a + 2b + c$.

Solution: To the first two expressions add the quantities obtained by changing the signs in the last two expressions.

$$\begin{array}{r}
2a + 7b - 15c \\
5a - 8b + 3c \\
-9a - 3b + 10c \\
4a - 2b - c \\
\hline
2a - 6b - 3c.
\end{array} \quad \textit{Ans.}$$

REMARK: The procedure used here is more efficient than adding the first two expressions, adding the last two, and then subtracting the second of these results from the first. The student should satisfy himself that the two methods must always yield the same answer. A numerical check may be made as in Example 1.

EXERCISES 4

Two operations are to be performed in the first six problems: (a) add the two expressions; (b) subtract the lower expression from the upper. Check the first eight problems by assigning numerical values to the letters, as in the preceding Example 1.

1. $\begin{array}{l} 5x - 3y + 6 \\ \underline{2x + y - 4} \end{array}$

2. $\begin{array}{l} 2a + 8b - 9 \\ \underline{3a - 4b + 11} \end{array}$

3. $\begin{array}{l} -7x + 2y - 3z \\ \underline{-2x - 3y - 9z} \end{array}$

4. $\begin{array}{l} 4r - 9s - 18t \\ \underline{8r - 2s - 13t} \end{array}$

5. $\begin{array}{r} 3u - 5v - 2w + 7 \\ \underline{-5u + v + w - 6} \end{array}$

6. $\begin{array}{l} 5e + 6f - 8g + 10 \\ \underline{4e + 7f + 8g - 10} \end{array}$

7. From the sum of $2x + y + 7z - 3$, $4x + 8y + 4z + 11$, and $3x + 4y - 7z + 6$ subtract the sum of $5x - y + 2z - 9$, $x + 10y - 3z + 17$, and $2x + 5y - 8z - 3$.

8. Subtract the sum of $4a + 6b + 2c + 7$, $5a - 2b - 9d + 3$, and $b + 2c - 3d - 5$ from the sum of $3a + 8b - 2c + d$, $2b - c - 3d - 5$, and $10a + 6b - 9c + 4$.

In Exercises 9 to 14 remove the signs of grouping and combine like terms·

9. $6x - (2x - 5y) - [3y - (5x + 2y)]$

10. $8y - (3x - 7y) - [4x - (3x - 4y)]$

11. $9s - \{6r - [(r - 4s) - (5r + 7s)]\}$

12. $12a - \{10b - [(4b - 3a) - (6b - 5a)]\}$

13. $[(d - e - 2f) - (3d + 2e + 5f)]$
$\qquad - [(5d + 4e - 3f) - (9d + e - 8f)]$

14. $[(7a + 2b - 3c) - (4a - 8b - 5c)]$
$\qquad - [(11b - 2a - 3c) - (4c + 2a - 6b)]$

For each example, 15 to 20, write an equivalent expression wherein the last three terms are enclosed in parentheses preceded by a minus sign.

15. $6 - 2x + y - 3z$

16. $11 - 4a - 7b - c$

17. $18 + u + 4v + 7w$

18. $-5 - r + 8s - 9t$

19. $-a - 4b + 3c - 6d$

20. $4e + 7f - 6k + 3h$

9. Exponents

If we should wish to indicate a multiplication involving the same factor several times, as $a \cdot a \cdot a \cdot a$, the writing of products would soon become too cumbersome for easy manipulation. We therefore define

$$a^2 \quad \text{as} \quad a \cdot a,$$

$$a^3 \quad \text{as} \quad a \cdot a \cdot a,$$

and, in general,

$$a^n \quad \text{as} \quad \underbrace{a \cdot a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}}.$$

The numbers that appear just to the right of and slightly above a in this definition are **exponents,** and they indicate the number of times that the *base* a is to be used as a factor; the exponents employed above are, respectively, 2, 3, and n. The symbol a^2 is usually read "a squared"; a^3 is "a cubed"; a^n is read "a to the nth power." When a letter appears with no exponent written, the exponent 1 must always be understood.

Illustration: Using exponents, $3a \cdot a \cdot a \cdot a \cdot b \cdot b \cdot b$ would be written $3a^4b^3$ and would be read "three a fourth b cubed."

Since an integral exponent indicates the number of times its base is used as a factor, we have, for example,

$$(a^2)(a^3) = (a \cdot a)(a \cdot a \cdot a) = a^5.$$

The result means that a is used as a factor five times in all. In general,

$$(a^m)(a^n) = \underbrace{(a \cdot a \cdot \cdots \cdot a)}_{m \text{ factors}}\underbrace{(a \cdot a \cdot \cdots \cdot a)}_{n \text{ factors}} = a^{m+n},$$

that is, a is used as a factor $m + n$ times in all.

This simple analysis gives us our most important rule of exponents, namely,

The Law of Exponents in Multiplication: *When two numbers having the same base are multiplied together, the result is a number with the same base and with an exponent which is the sum of the two exponents.*

Briefly, we say, "Add exponents in multiplication." The student should be warned that the full meaning of this rule is contained in the italicized statement. As a formula, we write

$$a^m \cdot a^n = a^{m+n}. \qquad (m \text{ and } n \text{ positive integers})$$

The discussion of exponents and the fundamental assumption that the order of factors in a product is immaterial furnish the basis for writing the product of two literal expressions.

To begin with, we consider one-term expressions which are also called **monomials.** Examples of monomials are $3a^2b^3$ and $-2ab^2$. In order to multiply these together, we take advantage of the commutative law of multiplication, and multiply first the numerical factors, then the parts involving a, and finally the parts involving b. Thus,

$$\begin{aligned}(3a^2b^3)(-2ab^2) &= (3)(-2)(a^2 \cdot a)(b^3 \cdot b^2)\\ &= (-6)a^{(2+1)}b^{(3+2)} = -6a^3b^5.\end{aligned}$$

When applying the rule of exponents to $a^2 \cdot a$, we must not fail to supply the exponent 1 for the second a.

This procedure applies equally well to the product of more than two factors. For example,

$$\begin{aligned}(2ab^2)(-a^3b)(4b^3) &= (2)(-1)(4)a^{(1+3)}b^{(2+1+3)}\\ &= -8a^4b^6.\end{aligned}$$

In these illustrations, the addition of the exponents has been written out in detail; the student should practice enough to be able to accomplish this addition mentally.

The evaluation of expressions involving exponents is illustrated in the next examples.

Example 1. Evaluate the expression $x^3 - 3x^2 + x + 2$ if x has the value -2.

Solution: The number -2 is substituted for x, and the required computations are carried out as follows:

$$\begin{aligned}x^3 - 3x^2 + x + 2 &= (-2)^3 - 3(-2)^2 + (-2) + 2\\ &= -8 - 12 - 2 + 2\\ &= -20. \quad \textit{Ans.}\end{aligned}$$

Remark: In replacing x by -2, we were careful to enclose the -2 in parentheses. This serves to warn us that each exponent applies to the *signed number* -2, not merely to its numerical value. The student must differentiate carefully between such items as $(-2)^2$ and -2^2. The first of these means $(-2)(-2) = +4$; whereas the second means $-(2)(2) = -4$. The point to be emphasized is that *an exponent applies only to the quantity that immediately precedes it.*

Example 2. Find the value of $2x + \frac{9}{x^2}$ when x has the value -4.

Solution: If $x = -4$,

$$2x + \frac{9}{x^2} = 2(-4) + \frac{9}{(-4)^2}$$

$$= -8 + \tfrac{9}{16}$$

$$= -7\tfrac{7}{16}. \quad \textit{Ans.}$$

EXERCISES 5

Find the value of each of the expressions in the first 15 problems.

1. $(-5)^2$ **2.** $(-3)^2$ **3.** $(-4)^3$
4. $(-2)^5$ **5.** $(\frac{1}{2})^3$ **6.** $(-\frac{1}{4})^3$
7. $(-\frac{2}{3})^4$ **8.** $(-2\frac{1}{3})^2$ **9.** $(-\frac{1}{2})^5$
10. $(2^2)(2^4)$ **11.** $(-3)^2(-3)^3$ **12.** $(-2)^3(-3)^2$
13. $(-3)(-2)^3(-5)^3$ **14.** $-(-7)^2(-1)^5$ **15.** $3(-2)^5(-1)^4$

In Exercises 16 to 27 simplify by applying the law of exponents in multiplication.

16. $x^2 \cdot x^7$ **17.** $y^3 \cdot y^6$ **18.** $a^3 \cdot a^7$
19. $u^3 \cdot u^m$ **20.** $r^m r^{2m+1}$ **21.** $(b^2)^3$
22. $(t^4)^5$ **23.** $(x^2y^2)^3$ **24.** $(a^3b^2)^4$
25. $(-3r^3s^5)^3$ **26.** $(-2c^6d^2)^4$ **27.** $-(-2x^2)^5$

Perform the following multiplications:

28. $3ab \cdot 6a^2b^4$ **29.** $7x^2y^3(-3xy^5)$
30. $2r^2s^3 \cdot rs^2(-3r^4s)$ **31.** $(6d^2e)(-3de^3)(-d^2e^4)$
32. $(-3x^2yz)(-2x^2y^2z^3)(-xy^3)^2$ **33.** $(-4a^2bc)^2(-2b^2c)^3(-c)^4$

Evaluate each of the following expressions if $a = 4$, $b = -3$, $c = -1$, $d = 5$, $x = 3$, and $y = -2$.

34. $(6 + y^2)(6 - y^2)$ **35.** $(4c + d)(10c^2 - d^2)$
36. $(a^2 + 3ab + b^2)(6c^2 - d^2)$ **37.** $(x^2 - x - 7)(x^2 + 2x - 5)$
38. $5 - \dfrac{6}{y^2} + \dfrac{1}{y^4}$ **39.** $\dfrac{2}{a^2} - \dfrac{3}{ab} + \dfrac{1}{b^2}$
40. $\dfrac{x^2 - 3xy - 4y^2}{8x^2 + 17xy + 2y^2}$ **41.** $\dfrac{3a^2 + ab - 2b^2}{3a^2 - 2ab - 8b^2}$
42. $\dfrac{a^2 + 3ay + 5y^2}{3a^2 - 7y^2}$ **43.** $\dfrac{3x^2 - cx - 8c^2}{x^2 + 2cx - c^2}$

10. Multiplication of Multinomials

The multiplication of two expressions of more than one term each is an extension of the work on multiplication of monomials. Expressions of more than one term are called **multinomials**; two-termed expressions are called **binomials** and three-termed expressions, **trinomials**. We shall use these last three names quite often.

Illustrations: $2a^2b + 3b^3$ is a binomial;

$x^2 + 3xy - y^2$ is a trinomial;

both of these, as well as expressions of more than three terms, are multinomials.

We can handle the multiplication of one multinomial by another by an almost direct application of the distributive law, that is, we may multiply either of the two expressions by the other, one term at a time, and add the results of this termwise procedure.

EXAMPLE 1. Multiply $(a^2 + 2ab - b^2)$ by $(a + 3b)$.

Solution: We use a vertical arrangement as follows:

$$\begin{array}{llll}
a^2 + 2ab - b^2 & & & \\
a \ + 3b & & & \\
\hline
a^3 + 2a^2b - ab^2 & & & \\
\quad\quad 3a^2b + 6ab^2 - 3b^3 & & & \\
\hline
a^3 + 5a^2b + 5ab^2 - 3b^3 & \textit{Ans.} & &
\end{array}$$

Observe that like terms are placed in the same column for convenience in addition.

Check: A worth-while check on the above result can be made by giving the letters numerical values and evaluating the factors and the product. For example, if $a = 2$ and $b = 3$,

$$a^2 + 2ab - b^2 = 7;$$

$$a + 3b = 11;$$

$$a^3 + 5a^2b + 5ab^2 - 3b^3 = 77 = (7)(11).$$

In making a check of this type, the numbers 1 and 0 should be avoided; these values will not check the exponents. It is not necessary to use large values for the letters. Although this evaluation does not furnish a perfect safeguard against error, it is quite unlikely that the numerical work will check when the algebraic work is wrong.

11. Division and Exponents in Division

By making use of the principle that *a number is unchanged in value if multiplied or divided by the number 1,* we may write

$$\frac{ab}{ac} = \frac{a}{a} \cdot \frac{b}{c} = \frac{b}{c}.$$

We make use of this idea in deriving

The Law of Exponents in Division:

(1) If m is greater than n $(m > n)$

$$\frac{a^m}{a^n} = \frac{\overbrace{(a \cdot a \cdot \quad \cdots \quad \cdot a)}^{(n \text{ factors})}\overbrace{(a \cdot a \cdot \quad \cdots \quad \cdot a)}^{(m - n \text{ factors})}}{\underbrace{(a \cdot a \cdot \quad \cdots \quad \cdot a)}_{(n \text{ factors})}} = a^{m-n} \qquad (a \neq 0)$$

(2) If $m = n$

$$\frac{a^m}{a^n} = \frac{a^m}{a^m} = 1 \qquad (a \neq 0)$$

(3) If m is less than n $(m < n)$

$$\frac{a^m}{a^n} = \frac{\overbrace{(a \cdot a \cdot \quad \cdots \quad \cdot a)}^{(m \text{ factors})}}{\underbrace{(a \cdot a \cdot \quad \cdots \quad \cdot a)}_{(m \text{ factors})}\underbrace{(a \cdot a \cdot \quad \cdots \quad \cdot a)}_{(n - m \text{ factors})}} = \frac{1}{a^{n-m}} \qquad (a \neq 0)$$

Illustrations:

$$\frac{a^6}{a^2} = a^{6-2} = a^4;$$

$$\frac{a^{10}}{a^{10}} = 1;$$

$$\frac{a^2}{a^5} = \frac{1}{a^{5-2}} = \frac{1}{a^3}.$$

For more complicated examples, the following should be studied with care.

Example 1. Simplify by removing the common factors in the division of $110a^3b^2c$ by $-154ab^4c$.

Solution:

$$\frac{110a^3b^2c}{-154ab^4c} = \frac{2 \cdot 5 \cdot 11a^3b^2c}{-2 \cdot 7 \cdot 11ab^4c}$$

$$= \frac{2 \cdot 11}{2 \cdot 11} \cdot \frac{5a^3b^2c}{-7ab^4c}$$

$$= -\frac{5}{7} \cdot \frac{a^3}{a} \cdot \frac{b^2}{b^4} \cdot \frac{c}{c}$$

$$= -\frac{5}{7} \cdot a^2 \cdot \frac{1}{b^2} \cdot 1$$

$$= -\frac{5a^2}{7b^2} \cdot \quad \textit{Ans.}$$

Observe that the numerical coefficients have been written as products of simple factors in order to aid in the simplification.

In forming the product

$$2x^2(5x^4 - 6x^2 + 17) = 10x^6 - 12x^4 + 34x^2,$$

we use the term-by-term procedure which has already been described. Since division is the inverse of multiplication, it appears that if we should divide $10x^6 - 12x^4 + 34x^2$ termwise by $2x^2$, the result must be $5x^4 - 6x^2 + 17$. Hence, *in order to divide a multinomial by a monomial, we divide each term of the dividend by the divisor and add the results.*

EXAMPLE 2. Divide $(x^2 + 2x - 3)$ by $\frac{1}{2}x$.

Solution: Divide each term of $x^2 + 2x - 3$ by $\frac{1}{2}x$.

$$\frac{x^2 + 2x - 3}{\frac{1}{2}x} = \frac{x^2}{\frac{1}{2}x} + \frac{2x}{\frac{1}{2}x} - \frac{3}{\frac{1}{2}x} = 2x + 4 - \frac{6}{x} \cdot \quad \textit{Ans.}$$

NOTE: When the division is not "exact," some terms of the result will remain in fractional form.

The student may easily verify that

$$\begin{aligned}(2x + 3)(3x - 5) &= 2x(3x - 5) + 3(3x - 5)\\ &= (6x^2 - 10x) + (9x - 15)\\ &= 6x^2 - x - 15.\end{aligned}$$

Now, if the final result and the factor $3x - 5$ were given, and we wished to find the other factor, this multiplication, so to speak, would have to be "undone." This means, of course, that we would have to divide $(6x^2 - x - 15)$ by $(3x - 5)$.

We argue that the first term of the dividend, that is, $6x^2$, came from multiplying the first term of the divisor, that is, $3x$, by $2x$. Hence, $2x$ must be the first term of the quotient, and it must be obtained by dividing $6x^2$ by $3x$. If we next multiply $3x - 5$ by $2x$ to obtain $6x^2 - 10x$ and subtract this product from the dividend, the result will be $9x - 15$ (see the second line of the multiplication above). Again, the first term of $9x - 15$, that is, $9x$, was found by multiplying $3x$ by 3. This means that the remaining term of the quotient can be found by dividing $9x$ by 3. The following form, which is modeled after ordinary arithmetic long division, will be found suitable for the preceding argument.

$$
\begin{array}{lrl}
 & & \overline{2x \; + \; 3} \qquad \text{(Quotient)} \\
 & \text{(Divisor)} \quad 3x - 5 & \big) \, 6x^2 - \quad x - 15 \quad \text{(Dividend)} \\
\text{Multiply divisor by } \dfrac{6x^2}{3x} = 2x & & \underline{6x^2 - 10x} \\
\text{Subtract; bring down next term, } -15 & & \qquad\quad 9x - 15 \\
\text{Multiply divisor by } \dfrac{9x}{3x} = 3 & & \qquad\quad \underline{\underline{9x - 15}}
\end{array}
$$

EXAMPLE 3. Divide $3 + 5c^3 - c^2$ by $c + 2$.

Solution: It is important in order to avoid confusion that dividend and divisor be arranged in order of decreasing exponents of c. It should also be noted that $0{\cdot}c$ has been written to preserve the space that would be occupied by the first power of c.

$$
\begin{array}{rl}
 & \overline{5c^2 - 11c \; + 22} \qquad \text{(Quotient)} \\
c + 2 \, \big) & 5c^3 - \quad c^2 + 0{\cdot}c + \; 3 \\
 & \underline{5c^3 + 10c^2} \\
 & \qquad - 11c^2 + 0{\cdot}c \\
 & \qquad \underline{- 11c^2 - 22c} \\
 & \qquad\qquad\quad + 22c + \; 3 \\
 & \qquad\qquad\quad \underline{+ 22c + 44} \\
 & \qquad\qquad\qquad\quad - 41 \qquad \text{(Remainder)}
\end{array}
$$

This means that we may write

$$\frac{5c^3 - c^2 + 3}{c + 2} = 5c^2 - 11c + 22 - \frac{41}{c + 2},$$

which is similar to the form used in arithmetic for an inexact division.

NOTE: It is generally desirable to arrange dividend and divisor in order of decreasing exponents of some letter as in Example 3. If more than one letter is involved, the arrangement may be in order of decreasing exponents of some particular one of the letters.

EXERCISES 6

Perform the following multiplications. Check your work by assigning convenient values other than 0 and 1 to the letters involved. Values that make the product 0 are also to be avoided.

1. $(x - 3)(x^2 + 4x - 7)$ **2.** $(2y + 5)(3y^2 - y + 4)$

3. $(z + 3)(z - 4)(z + 2)$ **4.** $(2 - c)(3 - 2c)(1 + 5c)$

5. $(a^2 + a - 6)(2a^2 - 3a + 1)$ **6.** $(3b^2 + 2b - 5)(2b^2 - b + 2)$

7. $(2x + y - 3z)^2$ **8.** $(u + 2v - 5w)^2$

9. $(x^2 + 4xy - 6y^2)(2x^2 - xy - y^2)$

10. $(2a^2 + ab + 4b^2)(3a^2 - ab + 2b^2)$

11. $(c - 2d + e - f)^2$

In Examples 12 to 20, inclusive, simplify by applying the laws of exponents. Each result should be written with the smallest possible exponents and numerical coefficients.

12. $\dfrac{105a^2b}{168a^5b^7}$ **13.** $\dfrac{20x^2y^7}{72x^5y^4}$ **14.** $\dfrac{56u^3v^5}{180u^7v^{10}}$

15. $\dfrac{48d^3e^5f^3}{-120de^7f^{10}}$ **16.** $\dfrac{198a^{10}b^{11}c^4}{60a^{12}b^7c^2}$ **17.** $\dfrac{84x^7y^6z^{15}}{63x^3y^{14}}$

18. $\left(\dfrac{-18c^4d^3}{30c^3d}\right)^2$ **19.** $\left(\dfrac{-35x^2y^5z}{14xy^7z^6}\right)^3$ **20.** $\left(\dfrac{-34a^5b^6c^8}{-51a^9b^2c^5}\right)^4$

In each example, 21–26, carry out the indicated division.

21. $\dfrac{8x^2 - 4y^2}{2xy}$ **22.** $\dfrac{6a^2b^4 + 18a^4b^2}{30ab^3}$

23. $\dfrac{c^4 + 4c^2d^2 + d^4}{c^2d^2}$ **24.** $\dfrac{x^8 - x^4y^6 + x^6y^4 - y^8}{x^4y^4}$

25. $\dfrac{5z^4 - 15z^3 + 35z^2 - 20}{10z^3}$ **26.** $\dfrac{a^6 + 3a^4b^2 + 3a^2b^4 + b^6}{12a^2b^2}$

Perform the indicated division in each of the following exercises. Check your work by multiplying the divisor by the quotient and adding the remainder.

27. $\dfrac{x^3 - 7x - 6}{x + 2}$ **28.** $\dfrac{y^3 - 2y^2 - 21y + 30}{y - 5}$

29. $\dfrac{16a^4 - b^4}{2a - b}$ **30.** $\dfrac{z^3 - z + 4}{z + 3}$

31. $\dfrac{2y^3 - 3y^2 + 7}{2y + 1}$

32. $\dfrac{5x + 3x^3 - 7}{x - 3}$

33. $(2m^3 + 7m^2 + 2m - 8) \div (2m^2 + 3m - 4)$

34. $(x^4 + 4) \div (x^2 - 2x + 2)$

35. $(12k^4 + 2k^3 + 9k^2 + 13k - 15) \div (4k^2 + 2k - 3)$

36. $(y^4 + 2y^3 - 12y^2 - 11y + 20) \div (y^2 - y - 5)$

37. $(x^3 - y^3 + 8z^3 + 6xyz) \div (x - y + 2z)$

38. $(27a^3 - 8b^3 - c^3 - 18abc) \div (3a - 2b - c)$

EXERCISES 7 · Review Problems

In each of the first five problems, (*a*) add the four expressions; (*b*) subtract the sum of the last three expressions from the first expression.

1. $4x^3 - 3x^2 - x + 7$, $2x^3 + 5x^2 + 6x - 4$, $-5x^3 - 2x^2 + 8x - 5$, and $x^3 + 6x^2 - 3x + 9$

2. $v^3 - 6v^2 + 2v$, $v^4 - 10v + 7$, $3v^4 + 2v^3 - 3v^2 - 10$, and $-2v^3 + 5v^2 - 11v + 12$

3. $5x - 2y + 3z$, $2 - 8x + 11y$, $13z + y + 5x$, and $7y - 6z - 14$

4. $12xy - 17yz + 15zx$, $16yz - 8xy + 11zx$, $-18zx + 3xy - 7yz$, and $4xy + 8yz - 21zx$

5. $23ab - 35ca + 10bc$, $16ca - 14bc - 32ab$, $-20ca + 27bc - 19ab$, and $42ab - 24bc - 17ca$

In Exercises 6 to 14, carry out the indicated multiplication.

6. $(a^4 - a^2b^2 + b^4)(a^2 + b^2)$

7. $(x^6 + 2x^3y^3 + 4y^6)(x^3 - 2y^3)$

8. $(9c^2 - 6cd^2 + 4d^4)(3c + 2d^2)$

9. $(1 - 5k^2 + 25k^4)(1 + 5k^2)$

10. $(3x^2 - x - 5)(2x^2 + 3x + 4)$

11. $(4s^2 + 2s - 7)(8s^2 - s - 2)$

12. $(t^2 + 4 - 9t)(7 + 6t - 3t^2)$

13. $(8u^2 + 6uw - 5w^2)(3u^2 - 2uw - w^2)$

14. $(7a^2 - 4b^2 + 9ab)(2ab - a^2 - 3b^2)$

In Exercises 15 to 18, perform the multiplications and collect like terms.

15. $(c - 5)^2 - (c + 5)^2$

16. $(x^4 + y^4)^2 - (x^4 - y^4)^2$

17. $(b^2 - 2b - 3)(b - 1) - (b^2 - b - 1)(b - 2)$

18. $(4a^2 + a - 7)(3a - 2) - (6a^2 - a - 3)(a - 5)$

In Exercises 19 to 26, perform the indicated division.

19. $(y^3 - y^2 - 4y - 6) \div (y - 3)$

20. $(15x^3 - 41x^2 + 26x - 28) \div (3x - 7)$

21. $(24a^2 + 18a - 7 + 15a^4 - 14a^3) \div (7 - 4a + 3a^2)$

22. $(2v^2 - 2v^3 + 8v^4 + 27v - 30) \div (3v + 4v^2 - 6)$

23. $(x^4 + x^2 + 1) \div (x^2 + x + 1)$

24. $(c^8 - 16d^8) \div (c^2 - 2d^2)$

25. $(3a^4b^8 - a^3b^6 - 4a^2b^4 + 32ab^2 - 24) \div (a^2b^4 - 2ab^2 + 4)$

26. $(27c^2x^4 - 15c^2x^2 + 45cx^2 - 12c^2 + 46c - 42) \div (4c - 6 + 9cx^2)$

Carry out the division in each of Exercises 27 to 32 to the point where the remainder contains no letter. Write each result in the form

$$\text{Quotient} + \frac{\text{remainder}}{\text{divisor}}.$$

27. $\dfrac{y + 3}{y - 1}$

28. $\dfrac{4x - 5}{4x - 3}$

29. $\dfrac{3v^2 - 8}{v - 2}$

30. $\dfrac{5a^2 + a - 3}{a + 5}$

31. $\dfrac{4c^2 + 7c - 11}{2c - 3}$

32. $\dfrac{6b^3 + 2b^2 - b + 13}{3b + 7}$

33. If eight packages are mailed for x cents each, eleven packages for y cents each, and six packages for z cents each, how much did it cost to mail the twenty-five packages?

34. If x dollars are invested at 3 per cent, simple interest, and y dollars are invested at 2 per cent, simple interest, what is the total amount earned in 1 year?

35. If A can do a job in a days and B can do a job in b days, what fractional part of the work is done if A works for m days and B works for n days?

36. Write the expression for the total distance driven by a motorist who travels for x hr at 50 mph, for y hr at 45 mph, and for z hr at 55 mph. What is the expression for the average speed?

37. A plane whose speed in still air would be r mph encounters a head wind of h mph. How far does the plane fly in n hr?

38. How far would the plane in the preceding problem fly in k hr if it had encountered a tail wind of v mph instead of a head wind?

Chapter 2

SIMPLE EQUATIONS, FORMULAS, AND STATEMENT PROBLEMS

12. Equations

A statement of equality between two quantities is called an **equation.** We have used such statements to a considerable extent in the preceding pages; for example,

$$a(b + c) = ab + ac$$

is an equation.

The last equation is called an "identity" because it is correct for any values that may be assigned to a, b, and c. An equation like $7 = 7$ is also called an identity.

On the other hand, the equation

$$x + 2 = 5$$

is not a correct statement unless x has the value 3. In other words, this statement imposes a condition on the value of x; x must be the number which when added to 2 gives 5. Any equation that imposes a condition on the letters involved is referred to as a **conditional equation.**

Conditional equations are widely used throughout mathematics and those sciences which are mathematical in character, such as physics, chemistry, and engineering. Every such equation says in effect, "Here is a condition which only certain numbers can satisfy. Find these numbers."

Much of algebra is devoted to this problem, and we shall investigate various means for solving it. For the present, we shall consider equations in which the value of only one letter is unknown; such equations

are frequently called **equations in one unknown.** In this chapter, moreover, we shall consider only equations which are given in (or can be changed to) a form that involves the unknown to no power higher than the first.

The two parts into which the sign of equality separates an equation are referred to as **sides,** or **members.** We shall use the terms **left side,** or **left member,** for the left-hand side and **right side,** or **right member,** for the right-hand side of an equation.

13. Solving Equations

If the members of a conditional equation can be made identical by replacing the unknown by a number, that number is called a **root** of the equation. The process of finding the roots of an equation is known as **solving** the equation.

Illustration: The number 9 is a root of the equation

$$x - 2 = 7;$$

for, if 9 is put in place of x, the obvious numerical identity $7 = 7$ results.

In order to solve a given equation, it is usually necessary to employ one or more of the fundamental operations. The use of these operations is discussed in the next section.

14. Operations That May be Performed on Equations

Before we begin the process of solving a conditional equation, we make the tacit assumption that there is a value of the unknown which makes both members of the equation identical. On the basis of this assumption, we employ operations that preserve equality and lead us to a proposed value of the unknown. The argument is capped by direct substitution into the original equation. If the latter is reduced to an identity, then our assumption and our procedure are both justified, and the proposed value is a root of the equation. On the other hand, if an identity does not result, then either an error has been made or else the given equation has no root.

It will be helpful to think of an ordinary beam balance in connection with the following discussion:

(1) ***Addition and Subtraction:*** *The same quantity may be added to or subtracted from both members of an equation.* This is the meaning of the usual brief statements: *Equals may be added to equals,* and *equals may be subtracted from equals.*

Illustrations: If to each side of the equation $5 = 5$ we add 2, the new equation $7 = 7$ is obtained. If 5 lb of sugar are placed in one pan of a balance, they will balance a 5-lb weight in the other pan. The addition of 2 lb of sugar to the first pan will be balanced by the addition of a 2-lb weight to the second pan. (Note that we are not implying that the sugar is the same as the weights. It is the equality of *weights* with which we are concerned, just as in the numerical example, it is the equality of the *numbers.*) It should be easy for the student to supply similar examples for subtraction.

EXAMPLE 1. Solve the equation $\frac{7}{4}y + 5 = \frac{3}{4}y + 1.$

Solution:

Subtract $\frac{3}{4}y$ from both sides $\quad y + 5 = 1.$

Subtract 5 from both sides $\quad y = -4.$ *Ans.*

Check: If $y = -4$

Left Member	Right Member
$= -7 + 5$	$= -3 + 1$
$= -2$	$= -2$

REMARK: The form given here is recommended for checking the solution of a given equation in order to avoid *assuming* that the answer checks before the verification is made. It also helps to prevent an invalid check in cases where complicated operations are performed on the equation. It should be emphasized that the substitution of a proposed root must be made in the members of the original equation, not in a derived equation.

(2) **Multiplication and Division:** *Both members of an equation may be multiplied or divided by the same quantity (zero excepted), provided that this quantity does not involve the unknown.* Again, this is the meaning of the briefer statement: *Equals may be multiplied or divided by equals.* Multiplication by zero will, of course, give equal members, but the equation will always be reduced to the trivial result $0 = 0$; division by zero is prohibited, as usual. Multiplication or division by a quantity involving the unknown is discussed in Section 26.

Illustrations: (*a*) If each side of the equality $5 = 5$ is multiplied by 2, the new equality $10 = 10$ is obtained. If the weight in one pan of a beam balance is doubled, the balance will be maintained if the weight in the other pan is also doubled. Similar illustrations may be supplied for the operation of division.

(*b*) The equation $2x = 10$ may be solved by dividing both members by 2 or by multiplying both by $\frac{1}{2}$. The result is $x = 5$.

If an equation in one letter can be reduced to the form $ax = b$ where a is not zero, the equation can always be solved by dividing both members by a or by multiplying by $\frac{1}{a}$.

EXAMPLE 2. Solve and check the equation

$$\tfrac{5}{8}v + 0.2 = \tfrac{3}{4}v + 0.6.$$

Solution: We subtract $\frac{3}{4}v$ and 0.2 from both sides of the equation to obtain

$$\tfrac{5}{8}v - \tfrac{3}{4}v = 0.6 - 0.2,$$

or

$$-\tfrac{1}{8}v = 0.4.$$

Next, we multiply both members by -8 to find

$$v = -3.2. \quad \textit{Ans.}$$

Check: If $v = -3.2$

Left Member	Right Member
$= -2.0 + 0.2$	$= -2.4 + 0.6$
$= -1.8$	$= -1.8$

Study the preceding problem with care. The principal idea has been first to reduce the equation to the form $ax = b$. In doing this, we employ addition, subtraction, multiplication, or division as may be desirable in each instance.

By a literal equation, we mean one which involves one or more literal numbers besides the one whose value is required. For example, $ax - 2b = c$ is a literal equation.

A root of an equation of this sort is usually not a specific number as in our previous problems but is instead a literal quantity. The steps used in solving such equations are identical with those previously employed; however, a major difference in detail appears because operations can frequently not be fully carried out but must be left in indicated form.

EXAMPLE 3. Solve $\quad ax - 2b = c \quad$ for $\quad x$.

Solution: We think of each letter except x in the equation as representing a known number.

Add $2b$ to both members $\qquad ax = 2b + c.$

Divide both members by a $\qquad x = \dfrac{2b + c}{a}\cdot$ *Ans.*

The check is left for the student.

EXERCISES 8

Show that the Equations 1 to 6 are identities.

1. $(a + 3)(a - 2) - 6 = (a + 4)(a - 3)$
2. $(x^2 + y^2)(x^4 - x^2y^2 + y^4) = x^6 + y^6$
3. $(3v + 11)(v - 2) = (v + 8)(v - 3) + 2v^2 + 2$
4. $c^2(c^2 + d^2) + d^4 = (c + d)^2(c - d)^2 + 3c^2d^2$
5. $\dfrac{s^3 - 7s + 6}{s - 1} = s(s + 1) - 6, \qquad s \neq 1$
6. $\dfrac{w^2 - 9}{w + 3} + \dfrac{w^2 - 16}{w - 4} = 2w + 1$, $w \neq -3$ and $w \neq 4$

Solve and check each of the following equations for the letter involved:

7. $0.8 - 6y = 4.8 - 7y$
8. $15.5 + 18x = 3.5 + 17x$
9. $\frac{5}{6}f - 0.6 = -\frac{1}{6}f - 3.4$
10. $\frac{17}{4}k + 8 = \frac{13}{4}k + 15$
11. $5(4t + 3) - 7(3t - 4) = 10$
12. $2(6h - 8) - 9(h + 4) = 2h$
13. $3w(w - 1) = 4w(w + 2) - (w^2 + 12w - 16)$
14. $6m(m + 4) + 3(6 - 2m^2) = 25m$
15. $(2x + 3)(5x - 1) = (3x + 2)^2 - (2 - x^2)$
16. $(4v + 5)(2v + 7) = (v + 2)(6v + 20) + (2v - 1)(v - 1)$
17. $\frac{1}{4}x - \frac{7}{12} = \frac{11}{12} - \frac{5}{4}x$
18. $\frac{2}{3}y - \frac{3}{5} = \frac{4}{3}y + \frac{2}{15}$
19. $\dfrac{2v + 7}{3} = 5$
20. $\dfrac{3w - 1}{7} = 2$
21. $\dfrac{10 - 3x}{5} = \dfrac{3}{2}$
22. $\dfrac{9 - 2t}{7} = \dfrac{4}{5}$
23. $\dfrac{7 - 4s}{3} = \dfrac{5 - 6s}{7}$
24. $\dfrac{2m - 17}{3} = \dfrac{8m + 11}{6}$
25. $\dfrac{8k + 3}{6} + \dfrac{1}{2} = \dfrac{7k - 1}{4}$
26. $\dfrac{4x - 23}{6} + \dfrac{1}{3} = \dfrac{5}{4}x$
27. $(6m - 1)(m - 7) - (2m - 3)(3m - 4) = 0$
28. $(5 - 3c)(7 + 4c) + (3 - 2c)(11 - 6c) = -14$
29. $(2v + 7)(4v - 5) - (6v + 11)(v - 4) = 2v^2 + 40$
30. $(x - 4)(3x + 10) - (2x - 5)(x - 6) = x^2 + 30$

15. Formulas

In algebra, a **formula** is a rule involving magnitudes and expressed in the form of an equation.

Illustration: Rule: To find the number of square units of area enclosed by a trapezoid, multiply the number of units of length in the altitude by half the number of units of length in the sum of the bases.

Let b_1 represent the number of units of length in one base; b_2, the number in the second base; h, the number in the altitude; and A, the number of units of area. Then, the formula

$$A = h\left(\frac{b_1 + b_2}{2}\right)$$

corresponds to the given rule.

NOTE: The numbers 1 and 2 in b_1 and b_2 are called "subscripts." b_1 is read, "b sub one"; b_2, "b sub two." The subscript notation is used frequently to distinguish two numbers when it is convenient to use the same letter for both. The student should be careful not to confuse subscripts and exponents.

The following formulas occur frequently in the geometrical applications of algebra. The student should study and learn them.

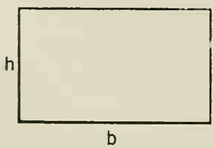


Fig. 5

(1) *Rectangle:* base b, altitude h (Figure 5).

Perimeter: $P = 2(b + h)$.

Area: $A = bh$.

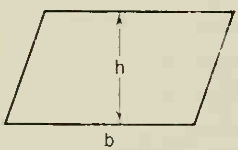


Fig. 6

(2) *Parallelogram:* base b, altitude h (Figure 6).

Area: $A = bh$.

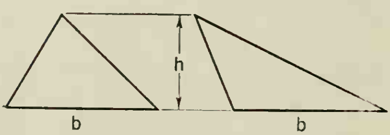


Fig. 7

(3) *Triangle:* base b, altitude h (Figure 7).

Area: $A = \frac{1}{2}bh$.

(4) *Trapezoid:* bases b_1 and b_2; altitude h (Figure 8).

Area: $A = \frac{1}{2}h(b_1 + b_2)$.

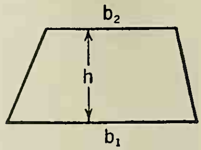


Fig. 8

(5) *Circle:* radius r (Figure 9).

Circumference: $C = 2\pi r$.

Area: $A = \pi r^2$.

($\pi = 3.1416$, correct to four decimal places.)

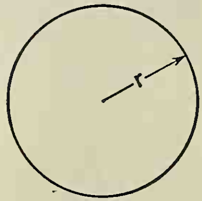


Fig. 9

(6) *Rectangular Box:* length l, width w, height h (Figure 10).

Volume: $V = lwh$.

Special Case: Cube of edge e:

$$V = e^3.$$

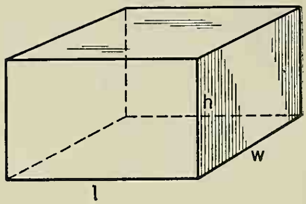


Fig. 10

(7) *Right Prism:* base area B, altitude h (Figure 11).

Volume: $V = Bh$.

(Bases may have any shape.)

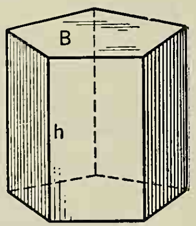


Fig. 11

(8) *Pyramid:* base area B, altitude h (Figure 12).

Volume: $V = \frac{1}{3}Bh$.

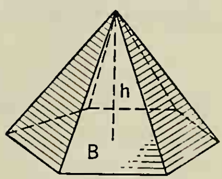


Fig. 12

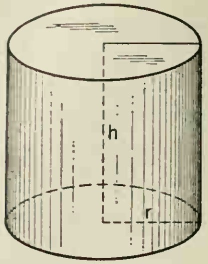


Fig. 13

(9) *Right Circular Cylinder:* radius r, altitude h (Figure 13).

Volume: $V = \pi r^2 h.$

Total Surface Area: $T = 2\pi r^2 + 2\pi rh.$

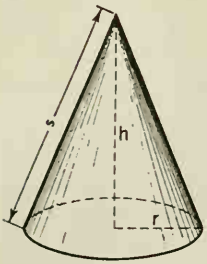


Fig. 14

(10) *Right Circular Cone:* radius r, altitude h, slant height s (Figure 14).

Volume: $V = \frac{1}{3}\pi r^2 h.$

Area of Curved Surface: $A = \pi rs.$

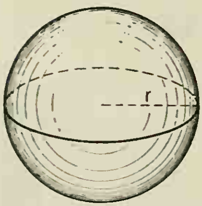


Fig. 15

(11) *Sphere:* radius r (Figure 15).

Volume: $V = \frac{4}{3}\pi r^3.$

Surface Area: $S = 4\pi r^2.$

There are many problems that require the solution of a formula for one of the letters involved. For example, if we wish to know how many years it will take an investment at simple interest to earn a given amount of interest I, we must solve the simple-interest formula

$$I = Prt$$

for t. This can be done by dividing both members by P and r. The result is

$$t = \frac{I}{Pr}.$$

In general, since formulas are merely literal equations, the methods employed upon such equations may be used. We shall illustrate this statement by an example.

EXAMPLE 1. The formula

$$A = P(1 + rt)$$

is used to find the total accumulation (principal plus interest) of an investment of P dollars at simple interest for t years if the interest rate is r. Solve this formula (*a*) for P; (*b*) for r.

Solution: (*a*) divide by $(1 + rt)$

$$\frac{A}{1 + rt} = P,$$

which means that $$P = \frac{A}{1 + rt}\cdot \quad Ans.$$

(*b*) Perform the multiplication in the right member of the given formula, thereby obtaining

$$A = P + Prt.$$

Subtract P from each member

$$A - P = Prt.$$

Divide by Pt $$\frac{A - P}{Pt} = \text{r},$$

that is, $$r = \frac{A - P}{Pt}\cdot \quad Ans.$$

NOTE: In changing to the final forms in (*a*) and (*b*) above, we simply read the previous equation in each case from right to left. The equations $x = a$ and $a = x$ say exactly the same thing.

EXERCISES 9

1. Find the area of a trapezoid whose bases are 6 ft and 18 ft, respectively, and whose altitude is 8 ft.

2. Calculate the volume and surface area of a cube of edge 7.6 in.

3. Find the area and circumference of a circle of radius 5.3 ft.

4. The circumference of a circle is 23π in. Find the radius and the area.

5. Calculate the surface area and the volume of a rectangular solid of length 7.9 ft, width 5.8 ft, and breadth 9 ft.

6. The base of a prism of altitude 12 in. is a right triangle whose legs are 5 in. and 12 in., respectively. What is the volume? What is the total surface area?

7. Find the volume and the lateral surface of a cylinder of radius 6 ft and altitude 9 ft. What is the total surface area?

8. The volume of a cylinder of radius 6 mm is 288π mm^3. What is the altitude? What is the total area?

9. Find the volume and the lateral surface of a cone of radius 16 cm and altitude 12 cm.

10. The volume of a pyramid with a square base is 132 ft^3. What is the altitude if the base is 6 ft on a side?

11. Find the volume and the surface area of a sphere of radius 9 in.

12. The base diameter and the volume of a cone are 10 ft and 100π ft^3, respectively. What is its altitude? its lateral surface?

Solve each equation for the letter, or each of the letters, listed after it.

13. $V = \pi r^2 h; \quad h$

14. $V = \frac{1}{3}\pi r^2 h; \quad h$

15. $s = \frac{1}{2}gt^2; \quad g$

16. $C = \frac{5}{9}(F - 32); \quad F$

17. $A = \pi r^2 + \pi rs; \quad s$

18. $s = \frac{1}{2}gt^2 + s_0; \quad g$

19. $l = a + (n - 1)d; \quad a, n, d$

20. $s = \dfrac{n}{2}(a + l); \quad a, n$

21. $F = k\dfrac{m_1 m_2}{d^2}; \quad d^2$

22. $A = \frac{1}{2}h(b_1 + b_2); \quad h, b_1, b_2$

23. $\dfrac{V_2}{V_1} = \dfrac{P_1}{P_2}; \quad V_1, P_1$

24. $F = \dfrac{Wv^2}{gr}; \quad r, W$

25. $s = \dfrac{a}{1 - r}; \quad a, r$

26. $I = \dfrac{E}{R + nr}; \quad R, n$

27. $\dfrac{1}{u} + \dfrac{1}{v} = \dfrac{1}{f}; \quad u, v, f$

28. $\dfrac{1}{R} = \dfrac{1}{r_1} + \dfrac{1}{r_2}; \quad R, r_1, r_2$

16. Statement Problems

In the applications of algebra, we frequently have occasion to solve problems that are first stated in words. If the problem describes a definite number, we may be able to translate the description into an equation whose solution is the number sought.

We shall consider a few problems that lead to equations of the first degree. The following examples should be examined with great care.

Example 1. How many ounces of a 50 per cent acetic acid solution should a photographer add to 32 oz of a 5 per cent acetic acid solution to obtain a 10 per cent solution?

Solution: In analyzing a problem of this sort, we must keep clearly

in mind that such a phrase as "a 5 per cent acetic acid solution" means that 0.05 of the total solution is acetic acid. In this case the 32 oz of 5 per cent solution contains (0.05)(32) or 1.6 oz of acetic acid. Hence, if we let

x = the *number* of ounces of 50 per cent solution to be added,

then, $0.50x$ = the *number* of ounces of acetic acid added,

and $1.6 + 0.50x$ = the *number* of ounces of acetic acid in the new solution.

$32 + x$ = the *number* of ounces of new solution of which 10 per cent is to be acetic acid.

Therefore, $$1.6 + 0.50x = 0.10(32 + x),$$

or $$1.6 + 0.50x = 3.2 + 0.10x.$$

$$0.40x = 1.6.$$

$$x = 4,$$

that is, 4 oz of the 50 per cent solution would be added. *Ans.*

Check:

$(0.50)(4) = 2$ (Ounces of acetic acid added);

$1.6 + 2 = 3.6$ (Ounces of acetic acid in new solution);

$32 + 4 = 36$ (Ounces of new solution);

$\dfrac{3.6}{36} = 0.10$ (New solution is a 10 per cent solution).

Comments: (*a*) The student should study a statement problem carefully in order to fix in his mind what conditions are given and what *numbers* are to be found.

(*b*) Having decided what is required, he should let x (or any convenient letter) stand for the *number of specified units* in one of the unknown quantities. It is important to keep in mind that x is a *number* and nothing else. Statements like, "Let x be the acid," are to be avoided; they do not make for clear thinking.

(*c*) Each unknown number involved in the problem should be expressed in terms of x. The preceding problem may serve as a model. Notice the statements which precede the equation.

(*d*) The equality stated or implied by the problem should be written as an equation in x.

(*e*) In a statement problem, the answer should be checked against the conditions of the problem. A check in the equation would be worthless if the equation were incorrect.

EXAMPLE 2. A man has 6 hr to spend on a hike into the mountains and back. He can walk up the trail at an average rate of 2 mph, while on the return trip, he can average 3 mph. How long a time may he spend on the up-trail hike if he plans to return on the same trail?

Solution: An *average rate of motion* is defined as *the total distance traveled divided by the time consumed.* The units of such a rate are always *units of distance per unit of time.* This fundamental definition must be borne in mind in connection with any motion problem.

Let t = the *number* of hours to allow for the up-trail hike,

then, $6 - t$ = the *number* of hours left for the return hike,

and $2t$ = the *number* of miles going up,

while $3(6 - t)$ = the number of miles coming down.

Since the distance up must be the same as the distance down

$$2t = 3(6 - t),$$

or

$$2t = 18 - 3t.$$

Therefore,

$$5t = 18,$$

and

$$t = 3.6.$$

Thus, he may spend 3.6 hr, or 3 hr, 36 min on the up-trail hike. *Ans.*

Check: $(2)(3.6) = 7.2$ (Miles, distance up trail);

$$\frac{7.2}{3} = 2.4 \quad \text{(Hours, time down)};$$

$$2.4 + 3.6 = 6 \quad \text{(Hours, total time)}.$$

EXAMPLE 3. A boy weighing 80 lb sits on a teeterboard 6 ft from the point of support and balances his brother who weighs 96 lb and sits 5 ft from the point of support. The lighter boy, by holding a weight in his lap, causes his brother to move back 6 in. to maintain the balance. How many pounds are in the weight?

Solution: The fundamental principle upon which we depend for the solution of a teeterboard problem is the *law of the lever*. Two weights on opposite sides of the point of support (fulcrum) will balance if, and only if, the product of one weight by its distance from the fulcrum is equal to the product of the other weight by its distance from the fulcrum; that is,

$$W_1d_1 = W_2d_2.$$

This neglects the weight of the lever. See Figure 16.

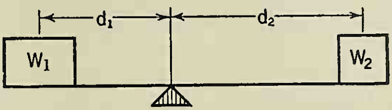


Fig. 16

Let w = the *number* of pounds in the unknown weight;

then, $w + 80$ = the *number* of pounds the heavier boy must balance.

Therefore, by the law of the lever,

$$6(w + 80) = (5.5)(96),$$

$$w + 80 = (5.5)(16),$$

or

$$w + 80 = 88,$$

and

$$w = 8.$$

There are 8 lb in the weight. *Ans.*

Check:

$(88)(6) = 528$ (*Wd* product for the 80-lb boy and weight);

$(96)(5.5) = 528$ (*Wd* product for the 96-lb boy).

Since these two products are the same, the answer, 8 lb, is correct.

EXERCISES 10

1. Find two numbers whose sum is 190 and such that one is three sevenths of the other.

2. Two numbers whose difference is 3 have a product which is 66 greater than the square of the smaller of the two numbers. Find the two numbers.

3. When each side of a square is increased by 7 in., the area is increased by 231 in.^{2} Find the side of the square.

4. What are the dimensions of a rectangle whose perimeter is 100 ft and whose length is 2 ft more than three times its width?

5. A picture without its border is 2 in. longer than it is wide. If the border is 1 in. wide and has an area of 40 sq in., find the dimensions of the picture alone.

6. Two spheres whose surface areas differ by 288π sq in. have radii which differ by 2 in. Find the radius of each sphere.

7. If the circumference of a ball is increased by 1 ft, by what amount is the radius increased? (NOTE: The results obtained indicate that the radius of an orange or of the earth would be increased by the same amount.)

8. The sum of the digits of a three-digit number is 16; the units' digit is 1 greater than the tens' digit, and the tens' digit is 3 greater than the hundreds' digit. Find the number.

9. The sum of the digits of a three-digit number is 19; the hundreds' digit is twice the tens' digit, and the units' digit is 3 greater than the tens' digit. Find the number.

10. In a three-digit number, the units' digit is double the tens' digit and the sum of the digits is 14. If the digits are reversed, the number is decreased by 396. Find the number.

11. In a Red Cross drive, A contributed twice as much as B, and C contributed \$4 more than A. How much did each give if their donations totaled \$79?

12. A fisherman caught 20 fish. If he had caught only 15, each fish would have had to average 2 oz more in order to give the same total weight. What was the weight of the catch?

13. A drugstore buys 18 cameras. If it had bought 10 cameras of a higher quality, it would have paid \$48 more per camera for the same total expenditure. Find the price of each type of camera.

14. A billfold contains \$223 in \$10, \$5, and \$1 bills. There are forty-seven bills in all and five more fives than tens. How many of each kind are there?

15. Two sums of money totaling \$25,000 earn, respectively, $2\frac{1}{2}$ per cent interest and 3 per cent interest per year. Find the two amounts if together they earn \$660.

16. The amount of annual interest earned by \$8,000 is \$40 less than that earned by \$12,000 at $\frac{1}{2}$ per cent less interest per year. What is the rate of interest on each amount?

17. A and B balance a teeterboard when A is 7 ft and B is 6 ft from the fulcrum. Find the weight of each if together they weigh 208 lb.

18. A teeterboard 12 ft long is balanced by A and B who weigh 91 lb and 117 lb, respectively. Where must the fulcrum be placed? Neglect the weight of the board.

19. A grocer mixes two kinds of nuts; one is worth 70 cents per pound, and the other is worth 90 cents per pound. If the mixture weighs 100 lb and is worth 81 cents per pound, how many pounds of each kind does he use?

20. How many gallons of a 15 per cent solution of salt must be added to 20 gallons of a 30 per cent solution to obtain a 27 per cent solution?

21. How many pounds of an alloy containing 35 per cent nickel must be melted with an alloy containing 65 per cent nickel to obtain 20 lb of an alloy containing 41 per cent nickel?

22. How many parts of a 30 per cent glycerin solution must be mixed with thirty parts of a 90 per cent glycerin solution to obtain a 75 per cent solution?

23. How many parts of glacial acetic acid (99.5%) must be added to 100 parts of a 10 per cent solution of acetic acid to give a 28% solution?

24. A jet plane flying at a speed of 600 mph is to overtake another plane which has a head start of $3\frac{1}{2}$ hr and is flying at a speed of 400 mph. How long will it take the faster plane to overtake the slower one?

25. Two planes which are 2475 miles apart and whose speeds differ by 75 mph fly toward one another and meet in 3 hr. What is the speed of each plane?

26. Find the hypotenuse of a right triangle if one of the legs is 20 ft long and the hypotenuse is 8 ft longer than the other leg.

27. One leg of a right triangle is 8 yd long, and the other leg is 2 yd shorter than the hypotenuse. Find the length of the hypotenuse.

28. How many minutes after 7 o'clock will the hands of a watch first be together?

29. At what time between 7 and 8 o'clock will the hands of a watch be opposite one another?

30. Crew A and crew B can do a job together in 6 days. If crew A can do it in 10 days, how long will it take crew B to do the job?

31. Crew A can do a job in 12 days and crew B can do a job in 6 days. How long will it take to do the job if both work at the same time?

Chapter 3

SPECIAL PRODUCTS AND FACTORING

17. Special Products

Certain of the results obtained in the multiplication of algebraic expressions are worth memorizing because of their frequent occurrence in algebra and its applications. The more important special products are given in the following list, each item of which may be verified by direct multiplication.

(1) *The square of the sum of two numbers is the sum of their squares plus twice their product:*

$$(a + b)^2 = a^2 + 2ab + b^2.$$

(2) *The square of the difference of two numbers is the sum of the squares of the two numbers minus twice their product:*

$$(a - b)^2 = a^2 - 2ab + b^2.$$

(3) *The product of the sum and the difference of two numbers is the difference of their squares (taken in the order in which the difference is given):*

$$(a + b)(a - b) = a^2 - b^2.$$

(4) *The square of a multinomial is the sum of the squares of its terms plus twice the product of each term and each term that follows it:*

$$(a + b - c)^2 = a^2 + b^2 + c^2 + 2ab - 2ac - 2bc.$$

(5) *The product of two binomials with a common term is the sum of three terms, namely,*

The square of the common term,

The product of the common term and the sum of the other two terms, The product of the two unlike terms:

$$(x + a)(x + b) = x^2 + (a + b)x + ab.$$

(6) *The product of two binomials with like terms is the sum of the products of the two like terms plus the sum of the products of the two unlike terms:*

$$(ax + by)(cx + dy) = acx^2 + (ad + bc)xy + bdy^2.$$

(7) *The cube of a binomial is the sum of the cubes of the two terms plus three times the square of each term multiplied by the other term:*

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,$$

and

$$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3.$$

EXAMPLE 1. Expand $(5x - 2y^2)^2$.

Solution: $(5x - 2y^2)^2 = (5x)^2 - 2(5x)(2y^2) + (2y^2)^2$

$= 25x^2 - 20xy^2 + 4y^4$. *Ans.*

EXAMPLE 2. Multiply $(2x + 3y)$ by $(2x - 3y)$.

Solution: $(2x + 3y)(2x - 3y) = (2x)^2 - (3y)^2$

$= 4x^2 - 9y^2$. *Ans.*

EXAMPLE 3. If $b + c = 10$, write out and simplify the product $(10a + b)(10a + c)$.

Solution: The "cross-product term" is $10a(b + c) = 100a$. Hence,

$$(10a + b)(10a + c) = 100a^2 + 100a + bc.$$

The first two terms are divisible by $100a$, so, if desirable, we may write

$$100a^2 + 100a = 100a(a + 1).$$

Consequently, our product may be expressed as

$$(10a + b)(10a + c) = 100a(a + 1) + bc. \quad \textit{Ans.}$$

NOTE: This example furnishes a neat little scheme for finding quickly the product of two numbers whose units' digits add to give 10 and whose other digits are identical. For example, by giving particular values to a, b, and c, the product of 97 and 93 may be written as follows:

$$a = 9; \quad a + 1 = 10; \quad a(a + 1) = 90;$$

$$b = 7; \qquad c = 3; \qquad bc = 21.$$

Therefore, $$(97)(93) = 9021.$$

The multiplication of $a(a + 1)$ by 100 moves this product to the left two places and leaves the bc product to fill the tens' and units' places. As another illustration, we may write the square of 115 by thinking

$$(11)(12) = 132 \quad \text{and} \quad 5^2 = 25,$$

so $$(115)^2 = 13{,}225.$$

Example 4. Expand the product $(8x - 3y)(5x + y)$.

Solution: To assist in obtaining the cross-product term, we note that

$$(8)(1) + (-3)(5) = 8 - 15 = -7.$$

Therefore, $(8x - 3y)(5x + y) = 40x^2 - 7xy - 3y^2$. *Ans.*

Note: When the binomials are written side by side, the following diagram is useful in obtaining the coefficient of the cross-product term:

$$(8x - 3y)(5x + y).$$

Example 5. Expand $(x - 2)^3$.

Solution: $$(x - 2)^3 = x^3 + 3x^2(-2) + 3x(-2)^2 + (-2)^3$$
$$= x^3 - 6x^2 + 12x - 8. \quad \textit{Ans.}$$

EXERCISES 11

Expand the following products:

1. $(v + 8)(v + 6)$
2. $(y - 9)(y + 6)$
3. $(2x + 5)(3x - 7)$
4. $(7z - 2)(3z - 1)$
5. $(3ab - 2)(4ab + 5)$
6. $(6b + 5k)(6b - 8k)$
7. $(8c^2 + 7)(8c^2 - 7)$
8. $(3r^2 + 5s^3)(3r^2 - 5s^3)$
9. $(3a - 5)^2$
10. $(2xy + 9)^2$
11. $(m^2 - 5)^3$
12. $(z^2 - 2u)^3$
13. $(8 - 3bc)(8 + 3bc)$
14. $(5 - 7c)(2 + 3c)$
15. $(9u^2 - 2v^4)^2$
16. $(4f - 5e)(5f + 4e)$
17. $(6x - 8h)(x - 7h)$
18. $(2y^3 - 11z)^2$
19. $(a + 4b + 5)(a + 4b - 5)$
20. $(x - 3y + 2)(x + 3y - 2)$
21. $(x - 2y - 2)^2$
22. $(r - 3s + 2t)^2$
23. $(4w^2 - 2k)(4w^2 - 11k)$
24. $(7 - a - 2b)(7 + a + 2b)$
25. $(r - 2s + t + 1)^2$
26. $(w - x - 2y - 2z)^2$
27. $[2a - 3(b + c)][a + (b + c)]$
28. $(3 - c^2 - d^2)(5 + 3c^2 + 3d^2)$

HINT: For Exercises 29 to 36, see the note in the preceding Example 3.

29. $(48)(42)$ **30.** $(74)(76)$ **31.** $(81)(89)$ **32.** $(53)(57)$
33. $(85)^2$ **34.** $(65)^2$ **35.** $(135)^2$ **36.** $(105)^2$

18. Factoring

We have previously stated that each of the numbers multiplied together to obtain a product is a factor of that product. There are many places in algebra where it is desirable, if possible, to break an algebraic expression down into an indicated product of its factors. This procedure is known as **factoring.** The same process takes place in arithmetic when we write

$$105 = (3)(5)(7).$$

A multinomial whose terms involve only positive integral exponents and do not involve division by a literal number is a **rational integral expression.** We shall limit our discussion here to the factoring of rational integral expressions with integral coefficients. In the remainder of this chapter, the term "expression" will be used for brevity to indicate this special type of multinomial. Furthermore, we shall consider only rational, integral factors with integral coefficients.

Any expression that is exactly divisible only by itself and the number 1 is called **prime.** *To factor an expression completely* will mean to write it as the product of some specific number and its prime factors.

Illustrations: $3a + 5b$ is a prime expression.

$3ab + 6ac$ is *not* a prime expression because it is exactly divisible by $3a$. If we write

$$3ab + 6ac = 3a(b + 2c),$$

we have factored the expression completely.

In order to develop a reasonable facility in factoring, we shall consider a number of type forms. As will be seen, the list of special products is to be considered of fundamental importance in algebra in the same way that the multiplication table is of importance in arithmetic.

In the expression $2ax - 6ay + 4az$, $2a$ is a factor of each term. Such a factor is known as a **common factor.** Its presence means that the expression is exactly divisible by $2a$, and we may write

$$2ax - 6ay + 4az = 2a(x - 3y + 2z).$$

EXAMPLE 1. Factor $5(a + b)x - 10(a + b)y$.

Solution: $5(a + b)$ is seen to be a common factor.

Hence, $5(a + b)x - 10(a + b)y = 5(a + b)(x - 2y)$. *Ans.*

Sometimes when no factor appears to be common to all the terms of an expression, the terms can be grouped so that each group has a common factor. This may enable us to factor the given expression.

EXAMPLE 2. Factor $ax + ay - az - 2bx - 2by + 2bz$.

Solution:
$$\begin{aligned} ax + ay - az - 2bx - 2by + 2bz &= (ax + ay - az) - (2bx + 2by - 2bz) \\ &= a(x + y - z) - 2b(x + y - z) \\ &= (x + y - z)(a - 2b). \quad \textit{Ans.} \end{aligned}$$

We see at once from the list of special products that the difference of two squares can be factored as follows:

$$\begin{aligned} 9x^2 - 16 &= (3x)^2 - (4)^2 \\ &= (3x + 4)(3x - 4). \end{aligned}$$

EXAMPLE 3. Factor $a^2 - b^2 + 2bc - c^2$.

Solution:
$$\begin{aligned} a^2 - b^2 + 2bc - c^2 &= a^2 - (b^2 - 2bc + c^2) \\ &= a^2 - (b - c)^2 \\ &= [a - (b - c)][a + (b - c)] \\ &= (a - b + c)(a + b - c). \quad \textit{Ans.} \end{aligned}$$

The idea of grouping terms is often useful in connection with the difference of two squares as may be seen from the preceding example.

EXERCISES 12

Each of the following is to be factored completely:

1. $5x^2y^3 + 30y^5$
2. $7a^2b - 42ab^2$
3. $14cd + 6ce + 2cf$
4. $36vx - 45vy - 24vt$
5. $15ab^2 - 20a^2b^3 - 25a^5b^7$
6. $44pt^5 - 8p^3t + 48p^2t^3w$
7. $25x^2 - 9y^4$
8. $49a^4 - 36b^{10}$
9. $16c^2d^6 - 81e^8$
10. $64s^6t^8 - 121x^2y^2$
11. $x^2 - 4x + 6xy - 24y$
12. $2w^2 + 3w - 10wb - 15b$
13. $24c^2 - 4r^2c^2 + 6m - mr^2$
14. $10a^2 - 15ae^2 - 2af + 3e^2f$
15. $8h^2k^5 - 18k^3$
16. $80m^2p^3 - 5p$
17. $x^4 - 16y^4$
18. $c^{16} - b^8$
19. $r^2 + 4rs + 4s^2 - 25$
20. $x^2 - 6xy + 9y^2 - 4$
21. $x^2 + y^2 - 1 + 2xy$
22. $9a^4 - 9 + 4b^2 - 12a^2b$

23. $bx^2 - 3bx + b - 2x^2 + 6x - 2$
24. $av - 2bv + 2cv - 7am + 14bm - 14cm$
25. $5sx - 5kx - 4sy + 15hx - 12hy + 4ky$
26. $20(c^3 + 2c^2d) - 24(cd + 2d^2)$
27. $35(x^4 - 3xy) - 15(x^3y - 3y^2)$
28. $x^2 + 4xy + 4y^2 - f^2 - 6fg - 9g^2$
29. $s^2 + t^2 - 25c^2 - 2st + 10cd - d^2$
30. $2a^{12}(3a^2 + 5)^9 - 6a^{14}(3a^2 + 5)^8$
31. $7x^6(2x^2 - 3)^6 + 21x^8(2x^2 - 3)^5$

Find the value of each of the following by factoring, and then multiplying the factors:

32. $(22)^2 - (19)^2$
33. $(48)^2 - (32)^2$
34. $(53)^2 - (43)^2$
35. $(74)^2 - (26)^2$
36. $(62)^2 - (67)^2$
37. $(75)^2 - (84)^2$

19. The Trinomials $x^2 + qx + r$ and $px^2 + qx + r$

An inspection of the list of special products shows that the multiplication of $(x + a)$ by $(x + b)$ gives the trinomial

$$x^2 + (a + b)x + ab.$$

We compare this expression with the trinomial

$$x^2 + qx + r.$$

As a result of the comparison, it is apparent that if we can find two numbers a and b whose product is r and whose sum is q, we may write

$$x^2 + qx + r = (x + a)(x + b).$$

The natural way to attempt to factor a trinomial of this type is to write r as the product of two integral factors in as many ways as possible. We may then examine these products to see if one of them consists of a pair of factors whose sum is q. If such a pair of factors exists, the factoring may be accomplished; otherwise, there are no factors with integral coefficients.

EXAMPLE 1. Factor $x^2 - 2x - 35$.

Solution: $-35 = (-1)(35) = (1)(-35) = (-5)(7) = (5)(-7)$. Of these, the first two may be immediately discarded. Why? The last two give for the sums of the factors

$$-5 + 7 = +2,$$

and $$5 - 7 = -2,$$

respectively. The last sum is the one we need. Thus,

$$x^2 - 2x - 35 = (x + 5)(x - 7). \quad \textit{Ans.}$$

Example 2. Show that $x^2 + 5x + 7$ is prime, that is, has no factors with integral coefficients other than itself and 1.

Solution: $7 = (1)(7) = (-1)(-7)$. The sum of neither of these pairs of factors is 5. Hence,

$$x^2 + 5x + 7 \text{ is prime.}$$

If we permit the coefficient of x^2 to be an integer other than 1, the process of factoring is usually more complicated. The problem is to find two first-degree factors whose product is the given trinomial. By direct multiplication, we have

$$(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd.$$

A comparison of this result with the trinomial

$$px^2 + qx + r$$

shows that we must find four numbers a, b, c, and d, such that

$$ac = p,$$

$$bd = r,$$

and $$ad + bc = q.$$

Example 3. Factor $5x^2 + 13xy - 6y^2$.

Solution: If this trinomial has factors of the type we seek, they must be of the form $ax + by$. As before, we shall work with the numerical coefficients only.

p	r	Cross Product
1 5	2 -3	$7 \neq q$
1 5	-3 2	$-13 \neq q$
1 5	3 -2	$13 = q$

Therefore, $5x^2 + 13xy - 6y^2 = (x + 3y)(5x - 2y)$. *Ans.*

The fact that -13 is obtained in the second trial suggests that the signs of the factors of r be reversed to give the desired cross product. A little practice should enable the student to discard certain combinations of factors mentally and to abbreviate the work considerably.

EXERCISES 13

Factor each of the following completely:

1. $x^2 + 8x + 12$

2. $y^2 - 3y - 10$

3. $24 - 2t - t^2$

4. $14 - 5x - x^2$

5. $3w^2 - w - 2$

6. $6d^2 - d - 15$

7. $9 - 6y + y^2$

8. $25 - 10b + b^2$

9. $v^2 - 9v + 18$

10. $r^2 - 11r + 30$

11. $3 + 5x - 2x^2$

12. $18 - 45x - 8x^2$

13. $z^4 - 18z^2 + 81$

14. $a^6 - a^3b^2 - 42b^4$

15. $9x^2 - 24xy + 16y^2$

16. $4c^4 - 28c^2d^3 + 49d^6$

17. $3y^2 + 6ay - 24a^2$

18. $5x^2 + 20ex - 105e^2$

19. $20v^2 - 23v + 6$

20. $12a^2 - 7a - 12$

21. $6m^2 - 73am + 12a^2$

22. $25z^4 - 25z^2d^2 + 4d^4$

23. $24b^2x^2 - 23abx - 12a^2$

24. $7e^2f^2 - 58efg + 16g^2$

25. $(a - b)^2 + 3(a - b) - 10$

26. $(3x - y)^2 + (3x - y) - 30$

27. $c^2 - 4cm + 4m^2 + c - 2m - 2$

28. $16a^2 - 8ab + b^2 - 12a + 3b - 4$

20. The Sum or Difference of Two Cubes

By direct division the student may show that $a^3 + b^3$ is exactly divisible by $a + b$, the quotient being $a^2 - ab + b^2$. Also, $a^3 - b^3$ is exactly divisible by $a - b$, the quotient being $a^2 + ab + b^2$. From these results, we have

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2),$$

and

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2).$$

EXAMPLE 1. Factor $27x^6 + 8$.

Solution: $27x^6 + 8 = (3x^2)^3 + 2^3$

$$= (3x^2 + 2)[(3x^2)^2 - 2(3x^2) + 4]$$

$$= (3x^2 + 2)(9x^4 - 6x^2 + 4). \quad \textit{Ans.}$$

21. Multinomials Which Can Be Made the Difference of Two Squares

Sometimes the device of adding and subtracting a perfect square may be used to change the form of a multinomial so that it may be written as the difference of two squares.

EXAMPLE 1. Factor $x^4 + x^2 + 1$.

Solution: By adding and subtracting x^2, we may write

$$
\begin{aligned}
x^4 + x^2 + 1 &= x^4 + 2x^2 + 1 - x^2 \\
&= (x^2 + 1)^2 - x^2 \\
&= (x^2 + 1 - x)(x^2 + 1 + x) \\
&= (x^2 - x + 1)(x^2 + x + 1). \quad \textit{Ans.}
\end{aligned}
$$

Two comments may be made here: First, the addition and subtraction of the same expression is a permissible operation. For, no matter what values may be assigned to the letters, this addition and subtraction has no effect on the value of the given expression. Second, that x^2 is the correct expression to add to form the perfect square may be seen as follows:

$$x^4 = (x^2)^2 \quad \text{and} \quad 1 = 1^2.$$

Hence, $2x^2$ is the correct cross-product term for the perfect trinomial square whose square terms are x^4 and 1, respectively.

EXERCISES 14

Factor completely:

1. $y^3 + 27$
2. $x^3 - 64$
3. $b^{12} - 125c^{15}$
4. $8a^9 + 27b^6$
5. $z^4 + 4$
6. $64e^8 + f^4$
7. $d^{12} + x^3y^3$
8. $64f^6g^{12} - h^6$
9. $a^4 - 3a^2 + 9$
10. $b^4 + 9b^2 + 25$
11. $8x^3 - y^3 - 2x + y$
12. $a^3 + 64b^3 - a^2 + 16b^2$
13. $4v^4 - 17v^2 + 4$
14. $c^8 + c^4 + 1$
15. $x^3 - 27y^6 + x^2 - 9y^4$
16. $25x^4 + 11x^2 + 4$
17. $(c - d)^3 - (a - 2b)^3$
18. $64a^6 - (b + 2c)^6$
19. $4(a + b)^4 + 1$
20. $r^6 - 3r^2 - 3t + t^3$

22. Summary of Factoring

The ability to recognize type forms is the key to skill in factoring. The following summary may prove helpful.

First, *any common monomial factor* should be removed. When this is done, the remaining expression may often be classified as one of the types below and then factored.

(1) *Binomials:*

(*a*) The difference of two squares.
(*b*) The sum or difference of two cubes.
(*c*) Reducible to the difference of two squares by the addition and subtraction of a perfect square.

(2) *Trinomials:*

(*a*) Perfect squares.
(*b*) Type $x^2 + (a + b)x + ab$.
(*c*) Type $acx^2 + (ad + bc)x + bd$.
(*d*) Reducible to the difference of two squares by the addition and subtraction of a perfect square.

(3) *Multinomials* (of more than three terms):

(*a*) Grouping to show a common multinomial factor.
(*b*) Grouping to form the difference of two squares.
(*c*) Cube of a binomial.
(*d*) Square of a multinomial.

EXERCISES 15

Factor each of the following completely:

1. $75 - 3y^2$
2. $54c^2 - 96d^4$
3. $2b^3 + 54$
4. $3h^2k^3 + 81h^2m^9$
5. $r^6 - r^5 - 12r^4$
6. $12y^4 - 7y^3 - 10y^2$
7. $21 - 4x - x^2$
8. $36 + 5s^2 - s^4$
9. $25m^2 - 40m + 16$
10. $9c^2 + 48c + 64$
11. $49 - 36a^2b^4$
12. $81x^4 - 16$
13. $12v^2 - 5v - 28$
14. $10u^2 + 3u - 18$
15. $d^{12} - c^{15}$
16. $x^6 + 27y^9$
17. $a^2 - 9b^2 - a - 3b$
18. $16v^2 - w^2 - 8v + 2w$
19. $64 - c^2 - 6cd - 9d^2$
20. $25 - 4x^2 + 4xy - y^2$
21. $e^4 + 7e^2 + 16$
22. $4 + y^{16}$
23. $a^3 - 5a^2 + 6a - 30$
24. $4w^3 - 6w^2 - 14w + 21$

25. $12x^4 - 7x^2 - 45$
26. $6u^4 - 31b^2u^2 + 28b^4$
27. $a^6b^6 - c^{12}$
28. $x^9 - y^9$
29. $m^3 - 2m^2 - 4m + 8$
30. $a^3 + 3a^2 - 9a - 27$
31. $b^3 - 12b^2 + 48b - 64$
32. $x^3 - 9x^2 + 27x - 27$
33. $16m^2 - y^2 + 8pm + p^2$
34. $4bc + a^2 - b^2 - 4c^2$
35. $5ur + 4 + 2u - 25r^2$
36. $6nu + 1 + 3n - 4u^2$
37. $16 - 16h^3 - 9s^2 + 9s^2h^3$
38. $9b^2x^6 + 4 - 4x^6 - 9b^2$
39. $(x^2 + 4x)^2 + 8(x^2 + 4x) + 16$
40. $(p^2 - 2p)^2 + 2(p^2 - 2p) + 1$
41. $(4y^2 + y - 5)^2 - (y^2 + 5)^2$
42. $(s^2 + 7)^2 - (3s^2 + 3s - 7)^2$
43. $2(x^3 + 1) + 5(x^2 - 1)$
44. $4(t^3 - 27) + 7(t^2 - 9)$
45. $(v^2 + v)^2 - 18(v^2 + v) + 72$
46. $(2m^2 - 3m)^2 - 11(2m^2 - 3m) + 18$
47. $2hk - ky - 3k + 8h - 4y - 12$
48. $12ex + 15x + 2f - 10 - 8e - 3fx$
49. $s^2 - 9 + t^2 - r^2 + 2ts + 6r$
50. $4x^2 + y^2 - w^2 - 4xy + 8w - 16$

Chapter **4**

FRACTIONS

23. Simplification, Multiplication, and Division of Fractions

In algebra, as in arithmetic, a **fraction** may be defined as an **indicated quotient.** The student will recall that the notation of fractions has already been used in connection with the division of algebraic expressions (see Chapter 1). Thus,

$$\frac{a+b}{c-d}$$

is an algebraic fraction; the expression $a + b$ is the numerator, and $c - d$ is the denominator. We shall assume throughout that the letters may be assigned no values that will involve a division by zero. The student may refer to page 6 for a discussion of division by zero.

The numerator and denominator of a fraction may be multiplied or divided by any number (except zero) without changing the value of the fraction. For example, we get

$$\tfrac{10}{12} = \tfrac{5}{6}$$

by dividing numerator and denominator by 2; also we obtain

$$\frac{ab}{ac} = \frac{b}{c}$$

by dividing numerator and denominator by a.

We shall define a **simple fraction** as one whose numerator and denominator are both rational integral expressions with integral coefficients. A simple fraction is said to be in its **lowest terms** when there are no factors other than 1 common to numerator and denominator.

The following examples show, in general, how we may *reduce* a fraction to its lowest terms. We factor the numerator and the denominator and divide both by their common factors.

Example 1. Reduce $\dfrac{30u^2v^2}{25u^2v - 20uv^2}$ to its lowest terms.

Solution:
$$\frac{30u^2v^2}{25u^2v - 20uv^2} = \frac{5 \cdot 6u^2v^2}{5uv(5u - 4v)}$$
$$= \frac{5}{5} \cdot \frac{u^2}{u} \cdot \frac{v^2}{v} \cdot \frac{6}{5u - 4v}$$
$$= 1 \cdot u \cdot v \cdot \frac{6}{5u - 4v}$$
$$= \frac{6uv}{5u - 4v}. \quad \textit{Ans.}$$

Example 2. Reduce $\dfrac{x^2 - a^2}{a^2 + ax - 2x^2}$ to its lowest terms.

Solution:
$$\frac{x^2 - a^2}{a^2 + ax - 2x^2} = \frac{(x - a)(x + a)}{(a - x)(a + 2x)}$$
$$= \frac{(x - a)(x + a)}{(-1)(x - a)(a + 2x)}$$
$$= -\frac{(x - a)}{(x - a)} \cdot \frac{(x + a)}{(a + 2x)}$$
$$= -\frac{x + a}{2x + a}. \quad \textit{Ans.}$$

In the last example, the factor $a - x$ in the denominator was written in the equivalent form $(-1)(x - a)$ in order to display the common factor $x - a$. The rule of signs in multiplication and division permits us to change the signs of an even number of factors in any multiplication or division. Thus, we may write

$$\frac{a}{b} = \frac{-a}{-b} = -\frac{a}{-b} = -\frac{-a}{b},$$

and
$$(a - b)(c - d) = (-1)(b - a)(-1)(d - c)$$
$$= (b - a)(d - c).$$

We recall from arithmetic that *the product of two fractions is formed by multiplying the numerator of one fraction by the numerator of the other, and similarly for the denominators.* Thus,

$$\left(\tfrac{2}{3}\right)\left(\tfrac{5}{7}\right) = \tfrac{10}{21}\cdot$$

Since we may consider an algebraic fraction as a general symbol standing for an arithmetic fraction, it follows that the same rule applies in finding the product of algebraic fractions.

Example 3. Form the product of $\dfrac{x^2y^2}{x^2 - y^2}$ and $\dfrac{x + y}{4xy}\cdot$

Solution:
$$\left(\frac{x^2y^2}{x^2 - y^2}\right)\left(\frac{x + y}{4xy}\right) = \frac{x^2y^2(x + y)}{(x - y)(x + y)(4xy)}$$
$$= \frac{x + y}{x + y}\cdot\frac{x^2}{x}\cdot\frac{y^2}{y}\cdot\frac{1}{4(x - y)}$$
$$= 1\cdot x\cdot y\cdot\frac{1}{4(x - y)}$$
$$= \frac{xy}{4(x - y)}\cdot \quad \textit{Ans.}$$

Note: In general, the product of two fractions should be reduced to its lowest terms as in the above example. Final denominators may be left in factored form.

We have already made use of the fact that division is the inverse of multiplication. This would mean that the result of dividing the fraction $\dfrac{N_1}{D_1}$ by the fraction $\dfrac{N_2}{D_2}$ is a quotient, say Q, such that

$$(Q)\left(\frac{N_2}{D_2}\right) = \frac{N_1}{D_1}\cdot \qquad \text{(Quotient times divisor equals dividend.)}$$

If we multiply both sides of this equation by $\dfrac{D_2}{N_2}$, we get

$$(Q)\left(\frac{N_2}{D_2}\right)\left(\frac{D_2}{N_2}\right) = \left(\frac{N_1}{D_1}\right)\left(\frac{D_2}{N_2}\right),$$

or

$$Q = \left(\frac{N_1}{D_1}\right)\left(\frac{D_2}{N_2}\right)\cdot$$

Now $\dfrac{D_2}{N_2}$ is the divisor *inverted;* hence, we have the rule: *The quotient of two fractions is found by multiplying the dividend by the inverted divisor.*

EXAMPLE 4. Divide $\dfrac{s^2 + 5s + 6}{2s^2 + s - 1}$ by $\dfrac{s^2 - 9}{(s + 1)^2}$.

Solution:
$$\frac{s^2 + 5s + 6}{2s^2 + s - 1} \div \frac{s^2 - 9}{(s + 1)^2}$$
$$= \frac{s^2 + 5s + 6}{2s^2 + s - 1} \cdot \frac{(s + 1)^2}{s^2 - 9}$$
$$= \frac{(s + 2)(s + 3)(s + 1)^2}{(2s - 1)(s + 1)(s - 3)(s + 3)}$$
$$= \frac{(s + 2)(s + 1)}{(2s - 1)(s - 3)}. \quad \textit{Ans.}$$

EXAMPLE 5. Simplify $\dfrac{a^2 - b^2}{3a + 3b} \div \left[\dfrac{2ab}{9a - 9b} \cdot \dfrac{(a - b)^2}{a^2b^2}\right]$.

Solution:
$$\frac{a^2 - b^2}{3a + 3b} \div \left[\frac{2ab}{9a - 9b} \cdot \frac{(a - b)^2}{a^2b^2}\right]$$
$$= \frac{a^2 - b^2}{3a + 3b} \cdot \frac{(9a - 9b)(a^2b^2)}{2ab(a - b)^2}$$
$$= \frac{(a - b)(a + b)(9)(a - b)(a^2b^2)}{3(a + b)(2ab)(a - b)^2}$$
$$= \frac{3ab}{2}. \quad \textit{Ans.}$$

NOTE: The brackets about the second pair of fractions in this example are necessary for clarity. The student may compare the meaning of $(4 \div 2)(3)$ with the meaning of $4 \div [(2)(3)]$ to understand this necessity. In all problems which involve combined operations, proper grouping symbols must be used to avoid misunderstanding.

EXERCISES 16

In each of the Problems 1 to 6, state the value or values of the letters or the relation between the letters for which the given fraction involves a division by zero and, hence, is meaningless.

1. $\dfrac{y - 5}{y - 3}$ 2. $\dfrac{a^2 + b^2}{5a + 10b}$ 3. $\dfrac{e^3 + 2}{e^2 - e - 12}$

4. $\dfrac{x^2}{x^2-9}$ 5. $\dfrac{v^2+vw}{v^2-6vw}$ 6. $\dfrac{m^2-rm}{m^2-9rm}$

Reduce each of the following fractions to its lowest terms:

7. $\dfrac{48a^2b^5c^7}{84a^2b^3c^6}$ 8. $\dfrac{60x^7yz^8}{84x^2y^5z^5}$ 9. $\dfrac{120r^8s^9t}{75r^3s^4t^5}$

10. $\dfrac{15a^7b^9}{21a^5b^4+27a^3b}$ 11. $\dfrac{18u^8v^6}{24u^5v^6-36u^6v^5}$

12. $\dfrac{36x^2-25y^2}{6x^2+5xy}$ 13. $\dfrac{c^3d^6+c^6d^3}{c^3+d^3}$

Perform the following multiplications and divisions and simplify the results:

14. $\dfrac{28a^2b^5c^8}{54r^4s^4t^7}\cdot\dfrac{18r^2st^3}{35ab^2c^3}$ 15. $\dfrac{40x^2yz^5}{33u^2v^8w^6}\cdot\dfrac{39u^4v^7w^9}{80x^3y^4z^3}$

16. $\dfrac{35ab^5}{24a^3b^9}\div\dfrac{15a^4b^7}{84a^3b^8}$ 17. $\dfrac{63st^8}{34s^4t^7}\div\dfrac{42s^6t^2}{17s^9t^3}$

18. $\dfrac{x^2+6x-16}{x^2-9x+14}\cdot\dfrac{x^2-8x+7}{x^2+10x+16}$ 19. $\dfrac{v^2+v-12}{v^2-8v+15}\cdot\dfrac{2v^2-7v-15}{3v^2+7v-20}$

20. $\dfrac{6-11w-10w^2}{2w^2+w-3}\div\dfrac{5w^3-2w^2}{3w^2-5w-2}$

21. $\dfrac{4b^2+12b+9}{9-4b^2}\div\dfrac{10b^2+27b+18}{8b^2-2b-15}$

22. $\dfrac{9s^2-6s+4}{25-4s^2}\cdot\dfrac{10+19s+6s^2}{27s^3+8}$ 23. $\dfrac{3x^2+11x+6}{4x^2+16x+7}\cdot\dfrac{28+x-2x^2}{2+x-3x^2}$

24. $\dfrac{(y+2)^4(y-2)^2}{(y^4-16)(y^2-4)}\div\dfrac{y^2+4y+4}{2y^3+8y}$

25. $\dfrac{z^2-3z}{z^2-3z+9}\div\dfrac{(z+3)^2(z-3)^3}{(z^3+27)(z^2-9)}$

26. $\dfrac{8x^2+2x-3}{3x^2+2x-8}\cdot\dfrac{3x^2-13x+12}{4x^2-17x-15}\cdot\dfrac{x^2-10x+25}{2x^2-7x+3}$

27. $\dfrac{e^2-6e}{6e^2-13e-63}\cdot\dfrac{18+5e-2e^2}{9h^2k^5}\cdot\dfrac{6h^4k^2}{12+4e-e^2}$

28. $\dfrac{c^3-d^3}{c^6-d^6}\cdot\dfrac{c^4-2c^2d^2+d^4}{c^2+cd+d^2}\cdot\dfrac{c^4+c^2d^2+d^4}{c^2-2cd+d^2}$

29. $\dfrac{y^2-(z-2x)^2}{(y-2x)^2-z^2}\cdot\dfrac{z^2-(2x-y)^2}{(z-y)^2-4x^2}\cdot\dfrac{4x^2-(y-z)^2}{(2x-z)^2-y^2}$

30. $\dfrac{v^3-w^3}{vw}\div\left[\dfrac{v^2+vw+w^2}{4v+4w}\cdot\dfrac{3v^2-3w^2}{v^2w^2}\right]$

31. $\dfrac{4z^2-7z-2}{6z^2+5z-6}\div\left[\dfrac{z^2-z-2}{3z^2-14z+8}\cdot\dfrac{4z^2-15z-4}{2z^2+5z+3}\right]$

24. Addition and Subtraction of Fractions

Two fractions are said to be *equivalent* if one can be made identical with the other by multiplying or dividing the numerator and denominator by the same number or expression. If zero denominators are avoided, two equivalent fractions always have the same value. Thus, we may write

$$\frac{2}{3} = \frac{4}{6}, \qquad \frac{x}{a} = \frac{2x}{2a},$$

and

$$\frac{x-a}{x+a} = \frac{(x-a)(x+a)}{(x+a)(x+a)} = \frac{x^2-a^2}{(x+a)^2}.$$

The addition of two fractions with the same denominator may be accomplished by a direct application of the distributive law of multiplication. For example,

$$\begin{aligned} \tfrac{2}{7} + \tfrac{3}{7} &= (2)(\tfrac{1}{7}) + (3)(\tfrac{1}{7}) \\ &= (2+3)\tfrac{1}{7} = \tfrac{5}{7}; \end{aligned}$$

and

$$\begin{aligned} \frac{a}{c} + \frac{b}{c} &= a\left(\frac{1}{c}\right) + b\left(\frac{1}{c}\right) \\ &= (a+b)\frac{1}{c} \\ &= \frac{a+b}{c}. \end{aligned}$$

This means that fractions with the same denominator are added by placing the sum of their numerators over the common denominator.

When the fractions do not have the same denominator, they should be changed to respectively equivalent fractions with a common denominator before the addition is performed. For example, to add $\frac{1}{3}$ and $\frac{1}{2}$, we write

$$\tfrac{1}{3} + \tfrac{1}{2} = \tfrac{2}{6} + \tfrac{3}{6} = \tfrac{5}{6}.$$

Similarly, to add $\dfrac{a}{x}$ and $\dfrac{y}{b}$, we write

$$\frac{a}{x} + \frac{b}{y} = \frac{ay}{xy} + \frac{bx}{xy} = \frac{ay+bx}{xy}.$$

In each of these illustrations, the common denominator has been taken as the product of the two denominators. If this procedure were

always followed, the result of the addition (or subtraction) would frequently not be in its lowest terms. As it is desirable to keep all fractions as simple as possible, we introduce the idea of the **least common denominator (LCD)**. *The LCD of a set of simple numerical fractions is the smallest positive integer that is exactly divisible by each denominator in the set.* (This number is also called the **least common multiple** of the denominators.)

EXAMPLE 1. Find the LCD of $\frac{1}{24}$, $\frac{1}{15}$, and $\frac{7}{20}$.

Solution: We may write each of the denominators in factored form.

$$24 = (2^3)(3);$$

$$15 = (3)(5);$$

and

$$20 = (2^2)(5).$$

In order to be divisible by 24, the LCD must have 2^3 and 3 as factors; to be divisible by 15, it must also have the factor 5. Since these three factors include the factors of 20, the LCD is $(2^3)(3)(5)$ or 120. *Ans.*

The corresponding discussion for algebraic fractions requires some preliminary definitions. *The degree of a monomial with respect to the letters involved is the sum of the exponents applied to its literal factors.* Thus, $125a^2b^3c$ is of degree 6 with respect to a, b, and c.

The degree of a rational integral expression with respect to the letters involved is the degree of the term (or terms) of highest degree. Thus, $5x^3 - 17x^2 + 3x - 1$ is of degree 3 with respect to x, and $u^3 - 5u^2v^2 + 3v$ is of degree 4 with respect to u and v together.

We may now define the **lowest common denominator (LCD)** for a set of simple algebraic fractions: *The LCD of a set of simple fractions is the expression of lowest degree with the smallest numerical coefficients that is exactly divisible by each denominator in the set.* (This expression is also called the **lowest common multiple** of the denominators.)

EXAMPLE 2. Find the LCD for a set of fractions having xy, x^2y^3z, $2x$, and $4z^2$ for denominators.

Solution: The separate factors present in these expressions are x, x^2, y, y^3, z, z^2, 2, and 4. In order to have an expression that is divisible by x^2, we must choose x^2 as a factor. This will make the expression automatically divisible by x. In the same way, we choose y^3 as a factor to give divisibility by both y^3 and y. z^2 will do for both z^2 and z, and the 4 will do for 4 and 2. Hence, the LCD is

$$4x^2y^3z^2. \quad Ans.$$

The examples that follow illustrate the rule: *To find the LCD of the denominators of a set of simple fractions:*

(1) *Factor each denominator completely.*

(2) *Write the product of all the distinct factors that occur, each with the greatest exponent applied to it anywhere in the given set of denominators.*

EXAMPLE 3. Find the LCD for the following set of denominators: $36x^2 - 36y^2$, $144(x^2 - xy - 2y^2)$, and $96(x + y)^2$.

Solution:

$$\begin{aligned} 36x^2 - 36y^2 &= (2^2)(3^2)(x + y)(x - y); \\ 144(x^2 - xy - 2y^2) &= (2^4)(3^2)(x + y)(x - 2y); \\ 96(x + y)^2 &= (2^5)(3)(x + y)^2. \end{aligned}$$

In accordance with the rule, the LCD is

$$(2^5)(3^2)(x + y)^2(x - y)(x - 2y),$$

or

$$288(x + y)^2(x - y)(x - 2y). \quad \textit{Ans.}$$

NOTE: It is usually convenient to leave the LCD in factored form.

The rule for the addition (or subtraction) of fractions may now be stated more explicitly: *In order to add (or subtract) a set of fractions:*

(1) *Change each fraction to an equivalent fraction with the LCD of the set as its denominator.*

(2) *Add (or subtract) the numerators of the equivalent fractions.*

(3) *Write the final result as the expression obtained in (2) divided by the LCD.*

EXAMPLE 4. Combine and simplify $\dfrac{1}{s - 2} + \dfrac{1}{1 - s} + \dfrac{s}{s^2 + 2s - 3}$.

Solution: Since $s^2 + 2s - 3 = (s - 1)(s + 3)$, and $\dfrac{1}{1 - s} = -\dfrac{1}{s - 1}$, the LCD is $(s - 1)(s - 2)(s + 3)$. Hence,

$$\begin{aligned} &\frac{1}{s - 2} + \frac{1}{1 - s} + \frac{s}{s^2 + 2s - 3} \\ &\qquad = \frac{(s - 1)(s + 3) - (s - 2)(s + 3) + s(s - 2)}{(s - 1)(s - 2)(s + 3)} \\ &\qquad = \frac{s^2 + 2s - 3 - s^2 - s + 6 + s^2 - 2s}{(s - 1)(s - 2)(s + 3)} \\ &\qquad = \frac{s^2 - s + 3}{(s - 1)(s - 2)(s + 3)}. \quad \textit{Ans.} \end{aligned}$$

The indicated sum or difference of a rational integral expression and one or more simple fractions is called a **mixed expression.** For instance,

$$a + b + \frac{1}{ab} \quad \text{and} \quad x^2 - y^2 + \frac{x+y}{x-y} + \frac{x-y}{x+y}$$

are mixed expressions. Such expressions arise in division problems, as we have previously seen. For example, by division, we find that

$$\frac{x^2+1}{x+1} = x - 1 + \frac{2}{x+1}.$$

The problem we wish to consider here is that of combining a mixed expression into a simple fraction. Since any expression divided by the number 1 is unchanged in value, we may think of the integral part of our mixed expression as a "fraction" with denominator 1. It is immediately apparent how to effect the desired combination.

Example 5. Write $3x - 1 + \dfrac{2}{x-1}$ as a simple fraction.

Solution:
$$\begin{aligned} 3x - 1 + \frac{2}{x-1} &= \frac{3x-1}{1} + \frac{2}{x-1} \\ &= \frac{(3x-1)(x-1)+2}{x-1} \\ &= \frac{3x^2-4x+3}{x-1}. \quad \textit{Ans.} \end{aligned}$$

EXERCISES 17

Combine each of the following expressions into a single fraction and simplify when possible:

1. $\dfrac{1}{28ab} + \dfrac{1}{36bc} + \dfrac{1}{21ca}$

2. $\dfrac{x}{12yz} - \dfrac{2y}{15zx} - \dfrac{3z}{20xy}$

3. $\dfrac{7}{60x^3y^2} + \dfrac{5}{36xy^5} - \dfrac{1}{225x^2y^6}$

4. $\dfrac{3}{16ab^3} - \dfrac{1}{36a^2b^5} + \dfrac{5}{48a^2b^7}$

5. $\dfrac{m}{20m-60} - \dfrac{5}{8m}$

6. $\dfrac{1}{18k} - \dfrac{5k}{12k+36}$

7. $1 + x + x^2 + \dfrac{x^3}{1-x}$

8. $1 - y^2 + y^4 - \dfrac{y^6}{1+y^2}$

9. $a^4 - a^2b^2 + b^4 - \dfrac{b^6}{a^2 + b^2}$

10. $9 + 27v^2 + \dfrac{81v^4}{1 - 3v^2}$

11. $\dfrac{2}{2x - 5} + \dfrac{x - 1}{x - 2} - 1$

12. $\dfrac{a + 2}{a - 3} - \dfrac{a - 1}{a + 3} + \dfrac{3}{4}$

13. $\dfrac{b}{b + 4} + \dfrac{b + 1}{2b - 8} + \dfrac{b^2 + 4}{16 - b^2}$

14. $\dfrac{3}{6 + x} + \dfrac{2x}{x^2 - 36} + \dfrac{5}{6 - x}$

15. $\dfrac{1 - c}{c^2 + 5c} + \dfrac{c - 2}{c^2 + 3c - 10} - \dfrac{7}{c}$

16. $\dfrac{5}{3 - 2v} + \dfrac{10v + 24}{6v^2 - 5v - 6} + \dfrac{4}{2 + 3v}$

17. $\dfrac{2}{1 - 2a} + \dfrac{6a + 18}{8a^2 + 6a - 5} + \dfrac{3}{5 + 4a}$

18. $\dfrac{2x - 3y}{3xy} - \dfrac{3y - 4z}{6yz} - \dfrac{2z - x}{2zx}$

19. $\dfrac{a - b}{2ab} - \dfrac{b + 2c}{4bc} + \dfrac{2c + a}{10ca}$

20. $\dfrac{s + 5}{s^2 - 5s + 4} + \dfrac{2s}{12 + s - s^2} + \dfrac{s + 7}{s^2 + 2s - 3}$

21. $\dfrac{2}{7 - 9x + 2x^2} - \dfrac{2x}{2 + x - 3x^2} - \dfrac{3x + 3}{6x^2 - 17x - 14}$

22. $\dfrac{a + 2}{9a^2 - 6a + 4} - \dfrac{14a - 6a^2}{27a^3 + 8} - \dfrac{1}{6a + 4}$

23. $\dfrac{y + 1}{3y^2 + 3y + 3} - \dfrac{8y^2 + 11y + 8}{12 - 12y^3} - \dfrac{3}{4y - 4}$

24. $\dfrac{v - x}{(y - x)(k - x)} - \dfrac{y - v}{(x - y)(k - y)}$

25. $\dfrac{1 + a}{b - b^2} - \dfrac{1 - a}{b^2 + b} - \dfrac{a^2 - 1}{b^2 - 1} + \dfrac{a^2}{b^2}$

26. $\dfrac{a}{(a - b)(c - a)} - \dfrac{b}{(b - c)(b - a)} + \dfrac{c}{(a - c)(c - b)}$

27. $\dfrac{r + s}{(t - s)(t - r)} + \dfrac{s + t}{(r - t)(r - s)} + \dfrac{t + r}{(s - r)(s - t)}$

28. $\dfrac{u^2 - (v - w)^2}{(w + u)^2 - v^2} + \dfrac{w^2 - (u - v)^2}{(v + w)^2 - u^2} + \dfrac{(w - u)^2 - v^2}{w^2 - (u + v)^2}$

25. Combined Operations

In any problem involving a combination of multiplication, division, addition, and subtraction, the student must pay careful attention to

grouping symbols. In general, the expressions that are grouped should be combined and simplified within their own parentheses before any attempt is made to combine different groups.

EXAMPLE 1. Simplify $\left(a - \frac{1}{a}\right) \div \left(1 + \frac{2}{a} + \frac{1}{a^2}\right)$.

Solution:
$$\left(a - \frac{1}{a}\right) \div \left(1 + \frac{2}{a} + \frac{1}{a^2}\right) = \frac{a^2 - 1}{a} \div \frac{a^2 + 2a + 1}{a^2}$$
$$= \frac{a^2 - 1}{a} \cdot \frac{a^2}{a^2 + 2a + 1}$$
$$= \frac{(a - 1)(a + 1)a^2}{a(a + 1)^2}$$
$$= \frac{a(a - 1)}{a + 1}. \quad \textit{Ans.}$$

EXAMPLE 2. Simplify $\left(\frac{1}{x - y} + \frac{1}{x + y}\right) \div \left(\frac{1}{x - y} - \frac{1}{x + y}\right)$

Solution:
$$\left(\frac{1}{x - y} + \frac{1}{x + y}\right) \div \left(\frac{1}{x - y} - \frac{1}{x + y}\right)$$
$$= \frac{x + y + x - y}{(x - y)(x + y)} \div \frac{x + y - x + y}{(x - y)(x + y)}$$
$$= \frac{2x}{(x - y)(x + y)} \cdot \frac{(x - y)(x + y)}{2y}$$
$$= \frac{x}{y}. \quad \textit{Ans.}$$

Sometimes, in place of using the division symbol as in the preceding example, we use the fraction bar as follows:

$$\frac{\dfrac{1}{x - y} + \dfrac{1}{x + y}}{\dfrac{1}{x - y} - \dfrac{1}{x + y}}.$$

A fraction of this type, which has simple fractions in its numerator and denominator, is referred to as a **complex fraction.** One way of simplifying a complex fraction is to rewrite it in the form corresponding to that in Example 2. The simplification would then be similar also.

A second method, which is frequently more efficient, is to multiply numerator and denominator of the complex fraction by the LCD of the simple fractions that are involved. This is illustrated by reworking Example 2 with the problem expressed in the form of a complex fraction.

The LCD of the simple fractions is $(x - y)(x + y)$; hence,

$$\frac{(x-y)(x+y)\left(\dfrac{1}{x-y} + \dfrac{1}{x+y}\right)}{(x-y)(x+y)\left(\dfrac{1}{x-y} - \dfrac{1}{x+y}\right)}$$

$$= \frac{(x+y) + (x-y)}{(x+y) - (x-y)} = \frac{2x}{2y} = \frac{x}{y}. \quad \textit{Ans.}$$

EXAMPLE 3. Simplify $\dfrac{1 + \dfrac{4uv}{(u-v)^2}}{1 + \dfrac{uv - 3v^2}{(u-v)^2}}$.

Solution: Multiply numerator and denominator of the complex fraction by $(u - v)^2$. There results

$$\frac{(u-v)^2 + 4uv}{(u-v)^2 + uv - 3v^2} = \frac{u^2 - 2uv + v^2 + 4uv}{u^2 - 2uv + v^2 + uv - 3v^2}$$

$$= \frac{u^2 + 2uv + v^2}{u^2 - uv - 2v^2}$$

$$= \frac{(u+v)^2}{(u+v)(u-2v)}$$

$$= \frac{u+v}{u-2v}. \quad \textit{Ans.}$$

EXERCISES 18

In each of the following exercises perform the indicated operations and simplify as much as possible:

1. $\dfrac{\dfrac{27a^3b^5}{7}}{9a^5b^2}$ **2.** $\dfrac{\dfrac{36u^5v}{5}}{4uv^5}$ **3.** $\dfrac{7x}{\dfrac{21x^3}{8x^4y}}$

4. $$\frac{\frac{1}{x^2}+\frac{3}{x}-4}{\frac{1}{x^2}+\frac{5}{x}+4}$$

5. $$\frac{\frac{6}{v^2}-\frac{11}{v}-10}{\frac{2}{v^2}+\frac{1}{v}-15}$$

6. $$\frac{y+3-\frac{16}{y+3}}{y-6+\frac{20}{y+3}}$$

7. $$\frac{w+2-\frac{18}{w-5}}{w-1-\frac{12}{w-5}}$$

8. $$\frac{\frac{8x}{3x+1}-\frac{3x-1}{x}}{\frac{x}{3x+1}-\frac{2x-2}{x}}$$

9. $$\frac{\frac{3}{m-4}-\frac{16}{m-3}}{\frac{2}{m-3}-\frac{15}{m+5}}$$

10. $$\frac{\frac{c}{d}-\frac{d}{c}}{\frac{c}{d}-2+\frac{d}{c}}$$

11. $$\frac{\frac{a^2}{b^2}+4+\frac{4b^2}{a^2}}{\frac{a}{b}+\frac{2b}{a}}$$

12. $$\frac{\frac{a^2-b^2}{a^2+b^2}-\frac{a^2+b^2}{a^2-b^2}}{\frac{a-b}{a+b}-\frac{a+b}{a-b}}$$

13. $$\frac{\frac{v+1}{v-1}+\frac{v-1}{v+1}}{\frac{v+1}{v-1}-\frac{v-1}{v+1}}$$

14. $$\frac{\frac{h}{h-1}-1-\frac{1}{h}-\frac{1}{h^2}}{\frac{1}{h^2-1}-\frac{1}{h-1}}$$

15. $$\frac{\frac{9}{y^2-9}+\frac{3}{y+3}}{1-\frac{3}{y}+\frac{9}{y^2}-\frac{y}{y+3}}$$

16. $$\frac{\frac{1}{xy}+\frac{2}{yz}+\frac{1}{zx}}{\frac{4x^2-(y+z)^2}{xyz}}$$

17. $$\frac{\frac{t}{rs}-\frac{r}{st}+\frac{2}{t}-\frac{s}{tr}}{\frac{1}{rs}-\frac{1}{st}+\frac{1}{tr}}$$

18. $$\frac{\frac{1}{c^2}-\frac{1}{4d^2}}{\frac{1}{c}+\frac{1}{2d}}-\frac{\frac{1}{c^3}-\frac{1}{8d^3}}{\frac{1}{c^2}-\frac{1}{4d^2}}$$

19. $$\frac{\frac{1}{25x^2}-\frac{1}{y^2}}{\frac{1}{5x}-\frac{1}{y}}+\frac{\frac{1}{x^2}-\frac{1}{25y^2}}{\frac{1}{x}+\frac{1}{5y}}$$

20. $$\frac{\left[\frac{a^2+b}{b^2-1}-\frac{1}{b-1}\right]-1}{\frac{(b-a)^2}{b+1}\left(\frac{a}{a-b}-1\right)}$$

21. $$\frac{\frac{1-4e^2}{(2e+f)^2}\left[1+\frac{f+1}{2e-1}\right]}{\frac{1}{2e+f}-\frac{1}{2e-f}+\frac{4e}{4e^2-f^2}}$$

22. $$\left(\frac{4-x}{x-1}-x\right)\left(\frac{x}{3x-2}+\frac{1}{x-2}\right)\div\left(3+\frac{24}{3x-2}\right)$$

23. $$\left(y-6+\frac{4}{y-2}\right)\div\left[\left(\frac{1}{y+2}-\frac{y}{7y-4}\right)\left(y-\frac{y+8}{y-1}\right)\right]$$

24. $$\left(2a-\frac{2ab+4a^2+b^2}{4a+b}\right)\left(8a+\frac{b^2}{2a+b}\right)\div\left(2+\frac{3b}{2a-b}\right)$$

25. $$\left(x+2y+\frac{8y^2}{x-2y}\right)\div\left[\left(2x-\frac{x^2}{x-2y}\right)\left(x+\frac{4y^2+4xy}{x-4y}\right)\right]$$

26. $\dfrac{2u+2}{u+1-\dfrac{u^2+u+1}{u+\dfrac{1}{u-1}}}$

27. $\dfrac{1-x^3}{1+\dfrac{x}{1-\dfrac{x}{1+x}}}$

26. Equations Involving Fractions

In order to simplify an equation which has fractions in its members, it is common practice to multiply both sides by the LCD of all the fractions involved. The result of this operation is an equation that is free of fractions. The methods previously studied may enable us to complete the solution of the equation.

Example 1. Solve the equation $\dfrac{1}{u}+\dfrac{1}{2}=\dfrac{5}{6u}+\dfrac{1}{3}\cdot$

Solution: The LCD of the fractions is $6u$. We multiply both sides by $6u$, and obtain

$$6u\left(\frac{1}{u}+\frac{1}{2}\right)=6u\left(\frac{5}{6u}+\frac{1}{3}\right);$$

$$6+3u=5+2u;$$

$$u=-1. \quad \textit{Ans.}$$

Check: If $u=-1$

Left Member	Right Member
$=\dfrac{1}{-1}+\dfrac{1}{2}$	$=\dfrac{5}{-6}+\dfrac{1}{3}$
$=-1+\dfrac{1}{2}$	$=-\dfrac{5}{6}+\dfrac{2}{6}$
$=-\dfrac{1}{2}$	$=-\dfrac{3}{6}=-\dfrac{1}{2}$

Example 2. Solve the equation $\dfrac{x^2}{x^2-1}=1+\dfrac{1}{x+1}\cdot$

Solution: The LCD of the fractions is $(x-1)(x+1)$. Both members of the equation are multiplied by this LCD to give

$$x^2=x^2-1+x-1.$$

$$-x=-2,$$

or $$x = 2. \quad \textit{Ans.}$$

The check is left for the student.

Remarks: If in the last example, a student should carelessly multiply both sides of the given equation by the product of the denominators, instead of the LCD, the result would be

$$x^2(x + 1) = (x + 1)(x^2 - 1) + (x^2 - 1),$$

or $$x^3 + x^2 = x^3 + x^2 - x - 1 + x^2 - 1.$$

Simplification of this equation gives

$$x^2 - x - 2 = 0,$$

or $$(x - 2)(x + 1) = 0.$$

This last equation will be satisfied by the values $x = 2$ and $x = -1$ as can be seen by substitution. This presents us with a difficulty because -1 is not a permissible value for x in the original equation; that is, its substitution would lead to a division by zero. We must agree, then, that the original equation and the last equation are not equivalent inasmuch as they do not have the same roots. This situation serves as a warning: *The solution of an equation is not complete until the original equation has been checked.*

The preceding remarks bring up two questions: (a) What are *equivalent* equations? (b) What operations performed on an equation will lead to an equivalent equation?

The first of these questions is answered by the following definition: *Two equations are equivalent if all the roots of one are roots of the other, and vice versa.*

Illustrations: The equations $x - 2 = 0$ and $x + 3 = 5$ are equivalent equations because the root of the first, $x = 2$, satisfies the second; and the second has no other root. On the other hand, the equation $(x - 1)(x - 2) = 0$ is not equivalent to either of the preceding equations; it has a root $x = 1$ which satisfies *neither* of the others even though $x = 2$ is a root common to all three.

The following statements pertaining to equivalence cover the four fundamental operations as we have used them:

(1) *If the same number or expression is added to or subtracted from both members of an equation, the result is an equivalent equation.*

(2) *If both members of an equation are multiplied or divided by any specific number except zero, the result is an equivalent equation.*

(3) *If both members of an equation are multiplied by an expression*

involving the unknown, the resulting equation may have roots that do not satisfy the original equation. If this happens, the new equation is sometimes called **redundant**; the roots that do not satisfy the original equation are called **extraneous.**

(4) *If both members of an equation are divided by an expression involving the unknown, the resulting equation may not be satisfied by all the roots of the original one.* In such a case, the new equation is frequently termed **defective.**

We may prove the first statement quite simply as follows: Let $A = B$ stand for the given equation, and let C be any number or expression. The equations

$$A = B$$

and

$$A + C = B + C$$

are equivalent. For, any root of the first equation will, upon substitution, reduce A and B to identical numbers and, hence, will reduce $A + C$ and $B + C$ to identical numbers. On the other hand, any root of the second equation will, upon substitution, reduce $A + C$ and $B + C$ to identical numbers and, therefore, will reduce A and B to identical numbers. This proof also covers the case of subtraction, since the sign of C was not specified and the subtraction of any number is equivalent to the addition of its negative.

The student may prove the second statement for himself by following the ideas in the preceding paragraph. The third and fourth statements are proved by the illustrations already presented. From the equation $x - 2 = 0$ we obtain $(x - 1)(x - 2) = 0$ by multiplication of both members by $x - 1$. The second equation is redundant, since the root $x = 1$ is extraneous to the given equation. On the other hand, the second equation being given, we may obtain the first by dividing both sides by $x - 1$. The equation $x - 2 = 0$ is defective with respect to the equation $(x - 1)(x - 2) = 0$; the root $x = 1$ is lost by dividing both sides by $(x - 1)$, an operation which would constitute a forbidden division by zero if x has the value 1.

The appearance of an extraneous root is not a very serious matter; for, as we have already seen, the final check will detect this situation. However, the check will not warn us in case a root is lost. This loss of roots may be avoided by the following procedure: If it is necessary to divide both members of an equation by an expression involving the unknown, this expression should be equated to zero in order to form an auxiliary equation. If any roots of the original equation have been lost, they will be found among the roots of the auxiliary equation. Hence,

all roots of the auxiliary equation should be checked in the given equation.

EXAMPLE 3. Solve the equation $\frac{1}{x} + \frac{1}{x-1} = \frac{1}{x(x-1)}$.

Solution: Multiply both sides by $x(x-1)$; this gives

$$x - 1 + x = 1;$$

$$2x = 2;$$

$$x = 1.$$

However, $x = 1$ is not a permissible value to substitute in the given equation. (Why?) Thus, we have an extraneous root only, and the given equation has no root.

EXERCISES 19

In Exercises 1 to 22, solve and check the equation for the letter involved.

1. $\frac{x}{6} - \frac{1}{2} = \frac{2}{3}$

2. $\frac{5y}{4} + \frac{3}{16} = \frac{1}{2}$

3. $\frac{k}{6} - \frac{k}{7} = \frac{1}{42}$

4. $\frac{4z}{3} - \frac{5z}{6} = \frac{3}{4}$

5. $\frac{x-2}{3} - \frac{x-3}{5} = \frac{13}{15}$

6. $\frac{8w+10}{5} + \frac{6w+1}{4} = 2w + 3$

7. $\frac{2}{3z} + \frac{1}{6z} = \frac{1}{4}$

8. $\frac{3}{8v} - \frac{1}{5v} = \frac{7}{10}$

9. $\frac{7}{w-6} = \frac{2}{w+4}$

10. $\frac{9}{5u-3} = \frac{5}{3u+7}$

11. $\frac{3x+4}{x-7} + 2 = \frac{2x+2}{x-4} + 3$

12. $2 + \frac{2-w}{w+2} = \frac{3w+7}{w+5} - 2$

13. $5\left(\frac{1}{2} + \frac{2u}{6-3u}\right) = 2\left(\frac{1}{4} - \frac{u-3}{3u-6}\right)$

14. $4\left(\frac{1}{6} + \frac{24-3x}{2x+15}\right) = 5\left(\frac{1}{3} - \frac{x-12}{2x+15}\right)$

15. $\frac{5}{2s+1} - \frac{2}{s-2} = \frac{1}{2s-1}$

16. $\frac{3}{y+4} - \frac{2}{y+2} = \frac{1}{y-2}$

17. $\frac{v+1}{v-4} - \frac{v}{v-2} = \frac{3}{v-6}$

18. $\frac{2}{4x+7} + \frac{x+2}{x+1} = \frac{2x+8}{2x+5}$

19. $\frac{5}{u^2-u-2} - \frac{3}{u^2+u-6} = \frac{4u+9}{(u+1)(u-2)(u+3)}$

20. $\dfrac{y+2}{y^2-1} - \dfrac{2y}{2y^2-y-3} = \dfrac{7(y-2)}{(y^2-1)(2y-3)}$

21. $\dfrac{4}{v-2} = \dfrac{5v}{v^2-4} - \dfrac{v+3}{v^2-2v}$

22. $\dfrac{2}{2t+3} = \dfrac{3t}{4t^2-9} + \dfrac{t-3}{4t^2+6t}$

In the remaining examples of this set, solve for the letter or letters listed after each equation.

23. $\dfrac{x}{a^2} - \dfrac{x}{b^2} = \dfrac{r}{s}\left(\dfrac{1}{a} - \dfrac{1}{b}\right)$; x

24. $\dfrac{1}{f}\left(\dfrac{1}{cv} - \dfrac{1}{dv}\right) = \dfrac{c}{d} - \dfrac{d}{c}$; v

25. $\dfrac{d+f}{fy-e} - \dfrac{d-f}{fy+e} = 0$; y

26. $\dfrac{a+b}{ex+n} - \dfrac{a-b}{ex-n} = 0$; x

27. $s = \dfrac{ra_n - a_1}{r-1}$; a_1, a_n

28. $E = RI + \dfrac{rI}{n}$; I, n, r

29. $Z_t = \dfrac{Z_1Z_2}{Z_1+Z_2}$; Z_1

30. $\dfrac{1}{s_1} - \dfrac{1}{s_2} = (u-1)\left(\dfrac{1}{r_1} - \dfrac{1}{r_2}\right)$; u

Show that each of the following equations has no root:

31. $\dfrac{3}{x+2} - \dfrac{1}{x-1} = \dfrac{x-4}{x^2+x-2}$

32. $\dfrac{5}{w-4} - \dfrac{2}{w+3} = \dfrac{w+17}{w^2-w-12}$

33. $\dfrac{z+3}{z-7} - \dfrac{z+1}{z+4} = \dfrac{8z-1}{z^2-3z-28}$

34. $\dfrac{7u+10}{6u^2-19u+10} = \dfrac{2u}{2u-5} - \dfrac{3u}{3u-2}$

35. $\dfrac{5}{2y^2+11y+12} + \dfrac{2}{y+4} = \dfrac{4}{2y+3}$

36. $\dfrac{4v-10}{3v^2+4v-15} + \dfrac{1}{v+3} = \dfrac{7}{3v-5}$

27. Problems That Lead to Equations Involving Fractions

The following examples illustrate further the use of our work with fractions.

EXAMPLE 1. A water tank can be filled by an intake pipe in 4 hr and can be emptied by a drain pipe in 5 hr. How long would it take to fill the tank with both pipes open?

Solution: Let t = the number of hours required to fill the tank with both pipes open.

The rate of flow for the intake pipe is one fourth of a tank per hour; for the drain pipe, the rate is one fifth of a tank per hour. Hence,

$\frac{t}{4}$ = the fraction of a tank that the intake pipe can fill in t hr;

$\frac{t}{5}$ = the fraction of a tank that the drain pipe can empty in the same time.

Since both pipes are to be open and *one* tank is to be filled, we must have

$$\frac{t}{4} - \frac{t}{5} = 1.$$

Therefore, $$5t - 4t = 20,$$

or $$t = 20. \quad \textit{Ans.}$$

Check: In 20 hr,
$$\text{Inflow} = (20)\tfrac{1}{4} = 5 \text{ tanks};$$
$$\text{Outflow} = (20)\tfrac{1}{5} = 4 \text{ tanks};$$
$$\text{Accumulation} = 5 - 4 = 1 \text{ tank}.$$

Therefore, 20 hr is the required time.

Example 2. It is known that the planet Venus makes a complete circuit about the sun in less time than the earth does. If Venus is between the earth and the sun once every 20 mo, how long does it take Venus to make a complete circuit about the sun? (Both planets travel about the sun in the same direction.)

Solution: Let n = the number of months it takes for Venus to make a circuit of the sun. Then $1/n$ is the fraction of a circuit that Venus makes in 1 mo. Since the earth makes one twelfth circuit in 1 mo,

$\frac{1}{n} - \frac{1}{12}$ = the fraction of a circuit that Venus gains on the earth in 1 mo.

But, according to the problem, Venus gains one twentieth circuit on the earth in 1 mo; hence,

$$\frac{1}{n} - \frac{1}{12} = \frac{1}{20}.$$

$$\frac{1}{n} = \frac{1}{20} + \frac{1}{12} = \frac{2}{15},$$

or $$n = \frac{15}{2} = 7\frac{1}{2}. \quad \textit{Ans.}$$

Check: If Venus makes a complete circuit of the sun in $\frac{15}{2}$ mo, the planet will make $\frac{2}{15}$ circuit in 1 mo, and will gain $\frac{2}{15} - \frac{1}{12} = \frac{3}{60} = \frac{1}{20}$ circuit on the earth in 1 mo, that is, Venus will be between the earth and the sun once every 20 mo. Therefore, $n = 7\frac{1}{2}$ mo is the required time.

EXERCISES 20

1. Find two numbers that differ by 7 and are such that one fifth of the smaller exceeds one ninth of the larger by 5.

2. Find two numbers that differ by 3 and are such that one seventh of the larger exceeds one twelfth of the smaller by 4.

3. The difference of two numbers is 63. If the larger is divided by the smaller, the quotient is 3 and the remainder is 9. What are the numbers?

4. The denominator of a fraction exceeds twice the numerator by 2. If the numerator is increased by 10 and the denominator by 24, the value of the fraction is unaltered. What is the fraction?

5. A grocer sells 15 lb of nuts at 75 cents per pound. How many pounds of nuts at $1.00 per pound must he sell to make his average price 85 cents?

6. A student has an average of 82 for 10 grades. How many grades of 90 must he receive in order to bring his average up to 85?

7. A manufacturer sold a quantity of pairs of skis for $1026; one fifth of them at $18 per pair, and the remainder at $24 per pair. How many of each kind did he sell?

8. It takes a plane flying 450 mph 25 min longer to go a certain distance than it does a second plane flying 500 mph. Find the distance.

9. A man drives a certain distance at the rate of 50 mph and a second man drives the same distance in 20 minutes less time at a rate of 60 mph. Find the distance.

10. Two cars have wheels of diameters 28 in. and 30 in., respectively. In what distance in feet would the smaller wheels make 40 revolutions more than the larger ones?

11. A man paid out one sixth of a cash fund for a boat and one fifth of the fund for an outboard motor. If $570 was left, how much was in the fund originally?

12. After depositing $1/a$ of his money at one bank and $1/b$ of his money at another bank, a man finds he has k dollars left. How much did he have at first?

13. The tens' digit of a two-digit number is 2 greater than the units' digit. If the number is divided by the sum of its digits, the quotient is 6 and the remainder is 1. Find the number.

14. If A can do a piece of work in r days, B in s days, and C in t days, how long will it take all three working together to do the work?

15. Three groups of men, A, B, and C, assemble ninety-six machines. A assembles three machines; B, four machines; and C, five machines per day.

If B works twice as many days as A, and C works one third as many days as both A and B together, how many days does each group work?

16. A bridge can be constructed by three groups of men. Group A can construct the bridge in 5 days, group B in 12 days, group C in 15 days. In how many days can group C finish the construction if group A works for 1 day and group B for 3 days?

17. A tank is filled by three pipes running simultaneously for 6 min. If the pipes were to run separately, it would take the second twice as long to fill the tank as the first, and the third three times as long as the first. How many minutes would it take each pipe alone to fill the tank?

18. A can do a piece of work in 9 days, B in 12 days, and C in 15 days. B works 3 days and stops. Then A works 3 days, after which C joins him. How long must A and C work together to finish the job?

19. A tank can be filled by one pipe in 16 min, by another pipe in 24 min, and can be drained by a third in 48 min. If all the pipes are open, in how many minutes can the tank be filled?

20. If in the preceding problem the drain pipe is closed after 6 min, how many more minutes would be required to fill the tank?

21. At a certain instant the sun, the earth, and the planet Mercury are in the same straight line. If Mercury requires 88 days to complete a circuit about the sun, when will the three next be in line? (Mercury and the earth travel their orbits in the same direction.) (NOTE: The planets may be on the same side or on opposite sides of the sun when the three bodies are in line.)

22. A pound of an iron-nickel alloy weighs 14.1 oz in water. When iron is weighed in water, it loses approximately one eighth of its weight; and nickel, one ninth of its weight. How many ounces of each metal are there in the alloy?

23. Gold loses approximately one nineteenth of its weight, and silver, approximately one tenth of its weight when weighed in water. An alloy of gold and silver weighs 45 oz in air and 42 oz in water. How many ounces of each metal are there in the alloy?

24. A brass (copper and zinc alloy) casting which weighs 20 lb in air loses 39 oz when weighed in water. Copper loses approximately one ninth of its weight, and zinc, one seventh of its weight when weighed in water. What is the percentage of copper in the brass?

25. A pound of solder composed of tin and lead weighs $14\frac{1}{4}$ oz in water. Tin loses approximately $\frac{5}{36}$ of its weight and lead approximately $\frac{5}{57}$ of its weight when weighed in water. Find the percentage of tin in the solder.

26. It would take group A 5 hr longer to assemble a machine than it would group B. After A works for as many hours as it would take B to do the entire job, B can finish the work in 3 hr. How long would it take each group alone to do the job?

27. A tank can be filled by one pipe in 7 hr more time than by a second pipe. If the smaller pipe is turned on for as many hours as it would take the larger pipe alone to fill the tank, the larger pipe can complete the filling in 5 hr. How long does it take each pipe separately to fill the tank?

Chapter 5

GRAPHICAL REPRESENTATION AND FUNCTIONAL NOTATION

28. Graphs

The pictorial representation of statistical data is so common that almost everyone has seen the numerous bar or circle graphs that appear in the newspapers and magazines. This method of presenting data is usually chosen because it shows in a vivid fashion such things as the progress of a bond drive, the number of automobile accidents caused by trucks contrasted with the number caused by pleasure cars, and so forth. A method of representation more important for us is presented in the following illustration.

Suppose that an engineer has to estimate the cost of making an open cut in a hill. Part of his problem is to estimate the area of a vertical section of the excavation. This he can do by making a survey of the section to find the height, or elevation, of points along the side of the hill in the section. The upper table on the opposite page might be the result of such a set of measurements.

Using the table, the engineer can draw a picture of the hill section in a perfectly natural manner (see Figure 17). On a sheet of paper ruled with vertical and horizontal lines, he draws a horizontal line on which to represent the floor of the excavation and a vertical line on which to represent elevations. These lines are **axes.** After choosing convenient units of length, he marks a scale along each line. The points on the side of the hill are then represented in an obvious fashion. These points have been circled in the figure. Observe that each point lies in the vertical line through the proper point on the horizontal axis and in the horizontal line through the proper point on the elevation axis. By connecting these points with a smooth curve, the engineer

has a fair approximation to the "profile" of the hill. The cross-sectional area can be found by a suitable area-measuring instrument (a planimeter) or by counting the number of squares in the area on the diagram. The curve which represents the profile of the hill is known as the **graph** of the elevation against the horizontal distance.

Vertical Section of Hill

Horizontal Distance from Foot of Hill (Ft)	Height of Hill above Floor Level of Excavation (Ft)
0	0
20	7
40	18
60	32
80	55
100	75
120	80

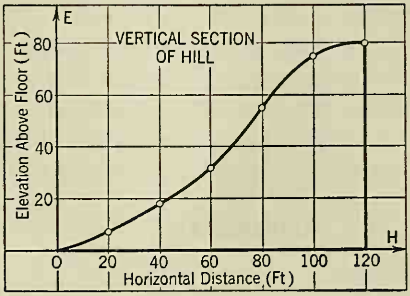


Fig. 17

In mathematics, the graph is of great importance in picturing a relation between two varying numbers. The procedure used by the engineer in this illustration is typical of that which we employ.

EXAMPLE 1. The following data are computed from an approximate formula which gives the minimum stopping distance D in feet from the point where the brakes are applied for an automobile traveling at a speed V miles per hour. These results are for a level, dry, concrete pavement and the best braking conditions for an average passenger car. Display the data graphically and read from the curve the stopping distance for a speed of 45 mph.

Speed (V Mph)	Stopping Distance (D Ft)
0	0
10	4.8
20	19.1
30	43.0
40	76.5
50	119.5
60	172.1

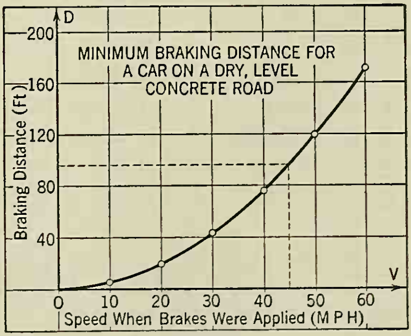


Fig. 18

Solution: We use the horizontal axis for plotting the speed V and the vertical axis for the stopping distance D. Suitable scales are chosen so that the curve will fill the available space to the best advantage. The points given by the table are plotted, using a value of V for the horizontal distance and the corresponding value of D for the vertical distance as may be checked in Figure 18. After all the points are plotted, a smooth curve is drawn through them. This curve is the required graph.

In order to find the stopping distance for a speed of 45 mph, first locate 45 on the horizontal axis. Then follow the vertical line through 45 to the point P on the curve, and from P proceed horizontally to the vertical axis, where 96 may be read. Thus, for a speed of 45 mph the stopping distance is approximately 96 ft.

EXERCISES 21

In each of the following problems, choose suitable scales and plot the data given in the table. Unless otherwise indicated, a smooth curve is to be drawn through the plotted points. The problems refer to their respective tables in these exercises. The data in the first column of each table are to be plotted along the horizontal axis. The axes should be labeled, and each graph should have a suitable title.

1. Table 1 gives the safe load T in tons for a manila rope of given diameter D in inches. What size rope is needed to support 9 tons?

Table 1

D	T
0.25	0.2
0.50	1.0
0.75	2.0
1.00	3.5
1.25	5.2
1.50	7.5
1.75	10.3
2.00	13.5
2.25	17.2
2.50	21.0

Table 2

H	E
0.2	21
1.0	46
2.0	66
3.0	74
4.0	78
5.0	80
6.0	81
7.0	80
8.0	79
9.0	78

Table 3

Date	M	Date	M
1800	5.3	1880	50.2
1810	7.2	1890	69.9
1820	9.6	1900	76.0
1830	12.9	1910	93.0
1840	17.1	1920	105.7
1850	23.2	1930	122.8
1860	31.4	1940	131.7
1870	38.6	1950	150.7

2. The horsepower output H and the corresponding efficiency E in per cent for an electric motor are given in Table 2. Read the efficiency for an output of 1.5 hp from your graph.

3. The population M of the United States in millions from the year 1800 to the year 1950 is given in Table 3.

4. An automobile was allowed to come to rest by the action of air resistance and frictional forces. The relation between the velocity v in miles per hour and the time t in seconds after the power was shut off is given in Table 4. In approximately how many seconds did the auto reach a speed of 40 mph? 25 mph?

5. Table 5 shows the relation between the cost C in dollars per kilowatt and the rated capacity K in kilowatts of a certain type of alternating-current generator. What would you estimate as the rated capacity of a generator of this type, costing $12 per kilowatt?

6. In a certain chemical reaction, a substance is used up as the reaction proceeds. Table 6 gives the percentage p of the substance remaining at the end of t min. What approximate percentage remains after 1 min? after 28 min?

7. The horsepower H of one type of water turbine for various values of the head of water h in feet is given in Table 7. Approximately what head of water is needed for 100 hp? A head of 6.2 ft will produce what horsepower?

Table 4

t	v
0	50
10	42
20	36
30	31
40	27
50	23
60	19
70	16
80	13

Table 5

K	C
50	20.0
75	17.0
100	15.0
150	13.5
200	12.5
300	11.4
500	11.0
750	10.8

Table 6

t	p
0	100
2	88
5	76
7	69
10	62
15	51
20	42
25	33
30	26

Table 7

h	H
4.0	66
4.5	80
5.0	93
5.5	107
6.0	123
6.5	140
7.0	159
7.5	184
8.0	216

8. For a certain electric generator the watt output W in watts and the corresponding total loss L in watts are given in Table 8. What is the approximate loss for an output of 1500 w? 2800 w?

9. One winter day, a thermometer recorded the air temperature F in degrees Fahrenheit for various times as shown in Table 9. In this problem, each plotted point should be joined to the next by a dotted straight line. We do this because the temperature may vary in a very irregular manner between successive points. The dotted lines serve only to carry the eye from one point to the next.

10. The distance s in feet through which a steel ball falls in t sec is given in Table 10. How many seconds does it take the ball to fall 80 ft? 160 ft?

11. In measuring the viscosity of extremely viscous liquids, use is made of a mobilometer. A measurement of the viscosity consists in finding the weight g in grams that will force a flat disk down through the liquid a dis-

tance of 10 cm in 100 sec. Table 11 gives the viscosity measurements for a heavy oil after the stated number h of hours of heat-treatment.

12. For men between the ages of fifteen and twenty-four, Table 12 gives the average weight w in pounds corresponding to a height h in feet and inches.

Table 8

W	L
250	920
1000	960
2000	1040
2500	1100
3000	1180
3500	1280
4000	1360
4500	1480
5000	1600
6000	1960

Table 9

t	F
8 A.M.	21
9 A.M.	22
10 A.M.	30
11 A.M.	46
12 M.	50
1 P.M.	54
2 P.M.	52
3 P.M.	42
4 P.M.	37
5 P.M.	26

Table 10

t	s
0.0	0
0.5	4
1.0	16
1.5	36
2.0	64
2.5	100
3.0	144
3.5	196
4.0	256
4.5	324

Table 11

h	g
0	1280
2	1390
4	1440
6	1440
8	1425
10	1410
12	1400
14	1390
16	1380
24	1350

Table 12

h	w
5′ 0″	120
5′ 1″	122
5′ 2″	124
5′ 3″	127
5′ 4″	130
5′ 5″	134
5′ 6″	138
5′ 7″	142
5′ 8″	146
5′ 9″	150
5′10″	154
5′11″	159
6′ 0″	165

Table 13

h	d
0.0	0.00122
0.5	0.00112
1.0	0.00104
1.5	0.00095
2.0	0.00088
2.5	0.00080
3.0	0.00074
3.5	0.00067
4.0	0.00061
4.5	0.00056
5.0	0.00052

Table 14

S	A
10	2.5
12	4
14	6
16	10
18	16
20	25
22	40
24	64
26	100
28	160
30	250
32	400

Table 15

Month	S
Jan.	15
Feb.	10
Mar.	13
Apr.	19
May	23
June	24
July	27
Aug.	18
Sept.	21
Oct.	20
Nov.	17
Dec.	16

13. Table 13 gives the relation between the altitude h in miles above sea level and the density d of the atmosphere in grams per cubic centimeter.

14. The exposure speeds for photographic films are measured by various light meters. The relation between the American Scheiner number S and

the ASA number A is given in Table 14. What would be the ASA reading for an American Scheiner reading of 25?

15. Table 15 shows the month-by-month sales S in thousands of units for a certain manufacturer for 1 yr. Join the points with dotted straight-line segments. Note that only the trend in a given portion of the curve can be observed and that no attempt should be made to read exact values other than the plotted ones from this curve.

29. Rectangular Coordinates

In order to obtain a graphical representation when the two related numbers may take on both positive and negative values, we use the conventions described below.

Draw two mutually perpendicular lines, one horizontal and one vertical (see Figure 19). These are the axes. The point of intersec-

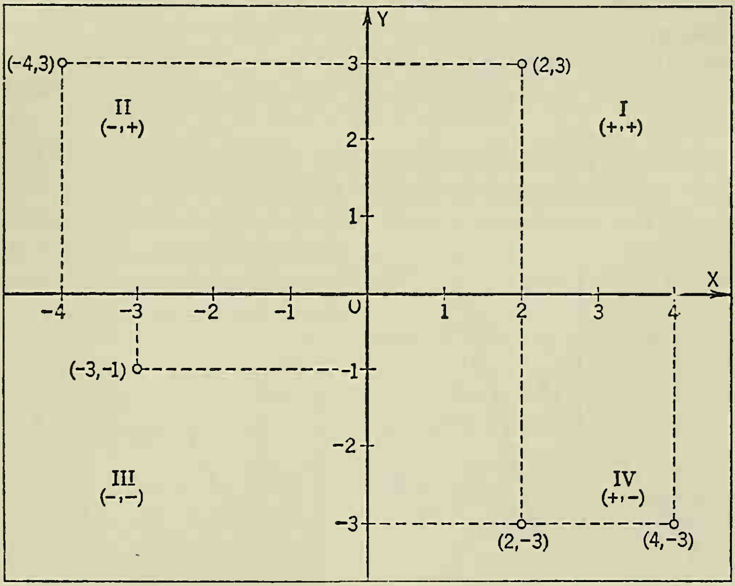


Fig. 19

tion, labeled O, is the zero point or **origin.** The horizontal line is the **X axis,** the vertical one, the **Y axis.** The *positive* directions on the axes are *to the right* and *upward,* respectively. The positive ends of the axes are labeled X for the horizontal and Y for the vertical one. A convenient unit of length is chosen for each axis (the two units need

not be the same), and points are marked on the axes with the signed numbers which they represent.

Any point in the plane can now be located by giving its distances from the two axes. These distances will in general bear the signs + or − to indicate the directions in which they are measured. Thus, a point whose horizontal distance is +2 and whose vertical distance is +3 will be located 2 units to the right of the Y axis and 3 units above the X axis; whereas a point whose horizontal distance is +2 and whose vertical distance is −3 will be located 2 units to the right of the Y axis and 3 units below the X axis.

For convenience in reference, we call the four portions into which the axes divide the plane **quadrants,** and number them around in counterclockwise order starting from the upper right quadrant as shown in the figure. The distances that we use to locate a point are termed **coordinates** of the point; the horizontal distance is the X **coordinate,** or **abscissa;** and the vertical distance is the Y **coordinate,** or **ordinate.** The coordinates of a point are usually given as an ordered pair of numbers, that is, in the form (horizontal distance, vertical distance) or (x, y).

We see now that each quadrant has a unique pair of signs to identify it; these are shown under the Roman numerals in Figure 19. Marking the point that represents a given pair of x and y values is known as **plotting the point.** There are several points plotted in the figure to illustrate the process.

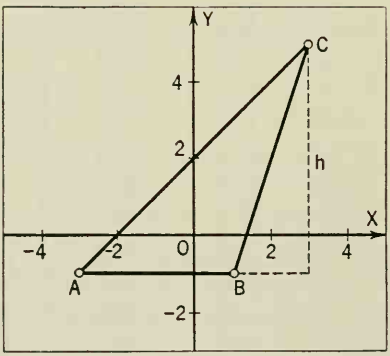


Fig. 20

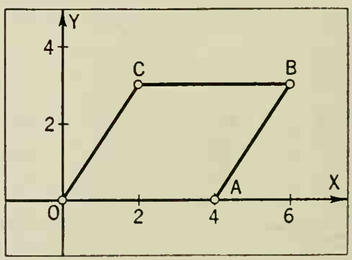


Fig. 21

Example 1. Plot the points $A(-3, -1)$, $B(1, -1)$, and $C(3, 5)$. Find the area of the triangle which has these points as vertices.

Solution: The points are plotted and the sides of the triangle are drawn as shown in Figure 20. In the triangle, the side AB which is

horizontal may be taken as the base; its length is 4 units. The altitude h is 6 units. Hence, the area of the triangle is

$$\tfrac{1}{2}bh = 12 \text{ square units. } \textit{Ans.}$$

EXAMPLE 2. Show that the points $O(0, 0)$, $A(4, 0)$, $B(6, 3)$, and $C(2, 3)$ are the vertices of a parallelogram.

Solution: The points are plotted and the connecting lines are drawn as shown in Figure 21. Since B and C have the same ordinate, namely, 3, the line CB is parallel to the X axis. Furthermore, OA and CB are each 4 units long. Therefore, the quadrilateral is a parallelogram. (Two of the sides are parallel and equal.)

EXERCISES 22

1. Plot the points (2, 6), (2, −1), (−5, 6), and (−5, −1). Show that these points are the vertices of a square.

2. Show that the points (4, 10), (4, −6), and (−2, 2) are the vertices of an isosceles triangle. Find its area.

3. Show that the points (2, 0), (12, 0), and (3, −3) are the vertices of a right triangle. Find its area.

4. The base of an isosceles trapezoid is the line that joins the points (3, −2) and (13, −2). If the third vertex is at the point (5, −6), what are the coordinates of the fourth vertex?

5. Three vertices of a parallelogram are (0, 0), (5, 0), and (4, 3). Find the coordinates of the fourth vertex if it is in (*a*) quadrant II; (*b*) quadrant IV; (*c*) quadrant I.

6. How far is it from the origin to each of the following points: (3, −4), (−12, −5), (8, 6), and (24, 7)?

7. How far is it from the point (3, 2) to the point (11, 8)?

8. Do the points (1, 3), (−1, −3), and (3, 9) lie on a straight line? Why?

9. Are the points (0, 1), (2, 5), and (3, 8) on a straight line? Why?

10. The point (−3, 5) bisects the line segment that joins the point (x, y) to the point (3, −1). Find x and y.

11. What must be the abscissa of any point on the Y axis? the ordinate of any point on the X axis?

12. What must be the relation between the two coordinates of any point on the line which bisects quadrants I and III? quadrants II and IV?

30. Graphs of Equations

The conventions of the preceding section may be used to make a graph of a given relation between two numbers. For example, if x and

y are a pair of numbers related by means of the equation $y = 2x - 1$, we may calculate values of y corresponding to given values of x. These pairs of values may be taken as the coordinates of points to be plotted. A smooth curve drawn through the points will be the graph of the relation defined by the equation. In the following table are given a few pairs of numbers; these are obtained by substituting convenient values for x and finding the corresponding values of y from the equation $y = 2x - 1$. In Figure 22, the points have been plotted and the graph has been drawn.

The student will notice that the graph appears to be a straight line; it is shown in analytic geometry that the graph of any equation of the form $ax + by + c = 0$, where a, b, and c are given constants, is a straight

$y = 2x - 1$

x	y
-2	-5
-1	-3
0	-1
1	1
2	3

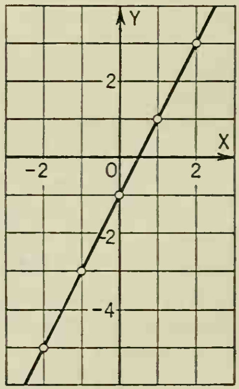


Fig. 22

line. For this reason, such an equation is called a "linear" equation. Since the equation $y = 2x - 1$ may be written in this general form, namely, $2x - y - 1 = 0$, it is a linear equation.

Since two points determine a straight line, the graph of a linear equation can be drawn conveniently by plotting two points on the line. However, the points should be taken as far apart as space on the graph will permit in order to draw the line as accurately as possible. It is also worthwhile to calculate and plot a third point as a check.

Example 1. Construct the graph of the equation $2x + 3y - 5 = 0$.

Solution: We solve the equation for y and find

$$y = \frac{5 - 2x}{3}.$$

Three points on the given line are $(-2, 3)$, $(1, 1)$, and $(4, -1)$. These are obtained by substituting each x value in the formula for y. The graph appears in Figure 23. Notice that the line crosses the X axis at the point where x has the value 2.5. This is the root of the equation $2x - 5 = 0$, because $y = 0$ for any point on the X axis.

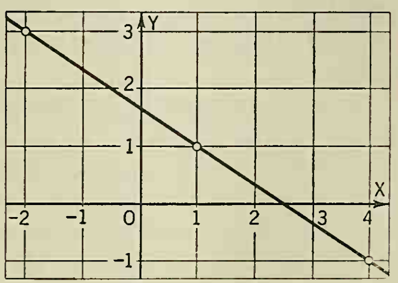


Fig. **23**

EXAMPLE 2. Plot the graph of the equation $y = x^2 - 4x + 1$. Use the graph to find approximations to the roots of the equation

$$x^2 - 4x + 1 = 0.$$

Solution: In the following table, which is constructed so that the details of calculation are as easy as possible, the items in each line of the two middle columns are added, and this result is increased by 1 to give the value of y. The student will find that such a table can be most rapidly constructed by completing one column at a time before passing to the next.

$y = x^2 - 4x + 1$

x	x^2	$-4x$	y
-1	1	4	6
0	0	0	1
1	1	-4	-2^*
2	4	-8	-3
3	9	-12	-2
4	16	-16	1^*
5	25	-20	6

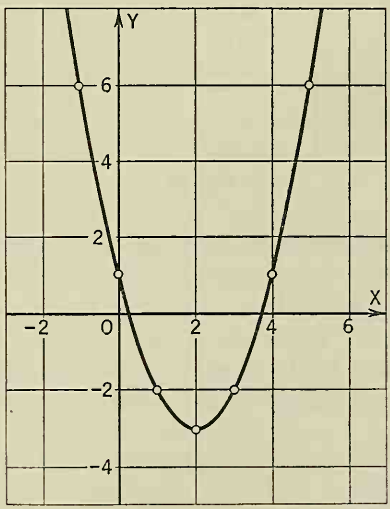


Fig. **24**

For convenience, we have calculated the values of y which correspond to integral values of x from -1 to $+5$. These values are put

into the table in the order that corresponds to moving from left to right along the X axis. This procedure should always be used as an aid to orderly thinking about the graph as it is being constructed. In the actual plotting (Figure 24), only the first and last columns of the table are used; the other columns have served their purpose in the preliminary calculation.

In order to solve the equation $x^2 - 4x + 1 = 0$, we have to find the values of x for which $y = 0$, that is, the abscissas of the points where the curve crosses the horizontal axis. For any point above this axis, the value of y is positive; whereas for any point below, the value of y is negative. Hence, we look for places in the table where there is a change in the sign of the ordinate. These places are marked with asterisks. It is clear from the graph that the curve crosses the X axis between the points where $x = 0$ and $x = 1$ and also between $x = 3$ and $x = 4$.

From Figure 24, we may read $x = 0.3$ and $x = 3.7$ as approximate roots of $x^2 - 4x + 1 = 0$.

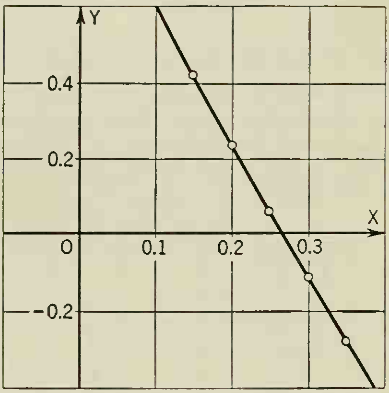


Fig. 25

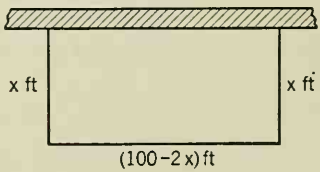


Fig. 26

If these roots were required more accurately, we could plot the graph on a larger scale in the neighborhood of the two values of x just found. This has been done in Figure 25 for the values $x = 0.15$, 0.2, 0.25, and 0.3. From this graph, we may read $x = 0.27$ as a closer approximation to the desired root. The same procedure may be used to find the root correct to any number of required digits.

Example 3. A farmer wishes to fence off a rectangular yard on one side of his barn. No fence is needed along the barn wall, and 100 ft of fencing is available for the other three sides. What is the largest area the yard can have?

Solution: Let x ft be the width of the yard. Then, $(100 - 2x)$ ft will be the length (see Figure 26). The number of square feet in the

area will be given by

$$\begin{aligned} A &= x(100 - 2x) \\ &= 2x(50 - x). \end{aligned}$$

In this problem, there is no practical significance attached to negative values of x or values of x greater than 50. Hence, we choose x values for our table from $x = 0$ to $x = 50$, with intervals of 5 for convenience.

Figure 27 shows the graph with A taken as the vertical coordinate and x as the horizontal coordinate. Since the largest value of A is wanted, we must first find the value of x which corresponds to the

$A = 2x(50 - x)$

x	$2x$	$50 - x$	A
0	0	50	0
5	10	45	450
10	20	40	800
15	30	35	1050
20	40	30	1200
25	50	25	1250
30	60	20	1200
35	70	15	1050
40	80	10	800
45	90	5	450
50	100	0	0

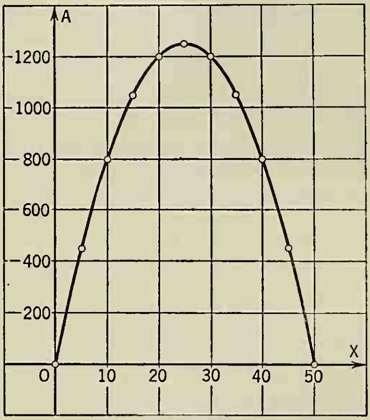


Fig. 27

highest point on the curve. It appears from the figure that $x = 25$ will give the largest value of A. This surmise can be verified by plotting more points in the neighborhood of $x = 25$. When $x = 25$,

$$A = 2x(50 - x) = (50)(25) = 1250.$$

The required maximum area is 1250 sq ft. *Ans.*

NOTE: A considerable amount of care must be exercised in choosing the origin and the scales for a graph. The origin should be chosen so that the graph will appear to good advantage in the available space. In general, the curve should be drawn to as large a scale as is compatible with the purpose of the drawing. A very large scale is necessary to show a small portion of the curve in detail, while a small scale may be needed to show the form of the entire curve. It is not necessary to use equal scales on both axes, except when it is desirable to show the curve in true

geometric proportion. For example, if the graph of a circle were plotted on axes with unequal scales, the curve would be distorted into an ellipse.

EXERCISES 23

Draw the graph of each of the linear equations in Problems 1 to 8. The lines may be drawn by first plotting two points, but a third point should be plotted as a check.

1. $3x - y = 0$

2. $x + y - 5 = 0$

3. $x + 2y + 6 = 0$

4. $2x + y + 8 = 0$

5. $4x + 3y - 15 = 0$

6. $3x - 2y - 16 = 0$

7. $6x - y = 14$

8. $5x + 16y = 20$

In each of Problems 9 to 22 make a table of values and plot enough points to show the general shape of the graph. Draw a smooth curve through the plotted points and read, correct to one decimal place, the roots of the equation obtained by putting $y = 0$. (HINT: Plot more points wherever you are not certain of the shape of the curve.)

9. $y = x^2 - 6x$

10. $y = 8x - x^2$

11. $y = x^2 + 3x - 4$

12. $y = 10 + 3x - x^2$

13. $2y = 4x^2 - 4x - 35$

14. $3y = 35 + 4x - 4x^2$

15. $y = x^3 - 9x$

16. $y = 9x^2 - x^3$

17. $y = x^3 + 4x^2 - 4x - 16$

18. $y = -9 + 9x + x^2 - x^3$

19. $y = x^2 - 4x + 2$

20. $y = 3 - x - x^2$

21. $y = x^3 + 2x - 7$

22. $y = x^3 - 5x + 1$

In the next four problems, the table of values may be constructed more easily by substituting values of y and calculating the corresponding values of x. Draw the graph and read, correct to one decimal place, the ordinates for which $x = 0$ in each case. Give the equation of which these ordinates are the roots.

23. $4x = 8 - 7y - y^2$

24. $x = 4y^2 - 4y - 15$

25. $x = 3 - 7y^2 - 2y^3$

26. $x = 6 + y^3$

Solve graphically the remaining problems in this set.

27. The sum of two numbers is 12. What are the numbers if their product is to be a maximum?

28. The difference of two numbers is 8. What are the numbers if their product is to be a minimum?

29. A rectangle has a perimeter of 28 in. What dimensions will give the greatest area?

30. An open trough is made by bending a long flat piece of tin 20 in. wide into a rectangular form. What must be the dimensions of the equal sides if the cross-sectional area is to be a maximum?

31. A farmer wishes to enclose a plot of ground in the shape of a rectangle. Only three sides are needed because the ground is adjacent to a long straight fence. What is the maximum area which he can enclose with an additional 400 yd of fencing?

32. A field is to be enclosed so that the area, rectangular in shape, is divided into five lots by fences running parallel to one of the sides. If 540 yd of fencing are available, what must be the dimensions of the field to obtain a maximum total area?

33. An open box is made by cutting small squares from a square piece of tin 12 in. on a side and then turning up the sides of the remaining piece. What size squares must be cut from the corners to form the open box with maximum volume?

34. A store owner finds that, if he can make 90 sales per day, he will make an average profit of 50 cents per sale. For every additional ten sales, his costs increase so that the profit on all sales is decreased by 2 cents per sale. How many sales a day will bring him the maximum total profit?

35. A manufacturer finds that for an output of 100 units of merchandise per week, he can make $50 profit on each unit. For every additional ten units manufactured per week, his costs increase so that the profit decreases by $1 per unit on the total output. Find the number of units per week that will bring him the maximum profit.

31. Functions and Functional Notation

In the preceding few pages, we have been making a graphical study of the behavior of related numbers. A large part of mathematics and its allied sciences is devoted to studying relations of this kind; hence, it is convenient to have a standard set of definitions and notations.

If a number, say y, *depends upon another number* x *in such a manner that when any permissible value is assigned to* x, *a corresponding value of* y *is determined, then* y *is called a function of* x.

Illustrations: (*a*) If $y = x^2 + 3x - 5$, the value of y is determined for any given value of x. We substitute the assigned value of x and perform the indicated operations to find the corresponding value of y. This means that y is a function of x.

(*b*) If all other conditions remain unchanged, the distance D in feet, in which a car can be stopped depends on the speed V in miles per hour at which the car was traveling when the brakes were applied. Here we say D is a function of V.

(*c*) The drop t in degrees in the freezing point of a salt solution depends on the strength of the solution, that is, the number x of grams of salt per liter of solution. Hence, t is a function of x.

In all these illustrations, the idea of functional dependence is the

same: The value of the function can be found (or is determined) when the value of the letter on which it depends is given. Notice that nothing has been said about the manner in which the value of the function is to be found. In many simple instances, the function is given by a formula; but in the second and third illustrations, it is conceivable that the value of the function could be found only by actual trial and measurement.

In the definition of a function, the number x, whose value we may assign, is frequently called the **independent variable.** The function y, whose value is then determined, is called the **dependent variable.** The term "variable" means that these letters, x and y, may represent various numbers in the course of a discussion of the function. Usually, other numbers whose values are fixed occur along with x and y. These fixed numbers are known as **constants.** The constants may be specific numbers, such as 2, 3, and $-\frac{1}{2}$ in the function

$$y = 2x^2 + 3x - \tfrac{1}{2};$$

or they may be literal numbers, whose values are thought of as fixed, such as a, b, and c in the function

$$y = ax^2 + bx + c.$$

We shall be concerned for the most part with relations involving only two variables, and it will be clear from the context which letters are variables and which are constants. Usually, the later letters of the alphabet are used for variables and the earlier letters for constants.

Often, when we wish to indicate that y is a function of x, we write

$$y = f(x).$$ (Read: "y equals a function of x.")

Observe carefully that the parentheses do NOT mean f times x! Other letters may be used instead of f to denote a functional relationship; frequently used are F, g, ϕ, and ψ.

When $f(x)$ is a given function, we may write, for example,

$$f(x) = 2x^2 + 3x - 1.$$

If we wish to indicate that a number, say 2, is to be substituted for x,we may conveniently write $f(2)$. Thus, if

$$f(x) = 2x^2 + 3x - 1,$$

then,
$$f(2) = 2(2)^2 + 3(2) - 1 = 13;$$
$$f(0) = 2(0)^2 + 3(0) - 1 = -1;$$
$$f(a) = 2a^2 + 3a - 1;$$

and $$f(x+h) = 2(x+h)^2 + 3(x+h) - 1.$$

If $f(x)$ is a given function of x, the symbol $f(a)$ means that x is to be replaced by a in the defining expression of the function.

Example 1. If $F(x) = \dfrac{1}{x}$, find $F(x+h) - F(x)$ in simplest form.

Solution: Substitute $(x+h)$ for x and obtain

$$F(x+h) = \frac{1}{x+h}.$$

We have $$F(x) = \frac{1}{x}.$$

Therefore, $$F(x+h) - F(x) = \frac{1}{x+h} - \frac{1}{x}$$

$$= \frac{-h}{x(x+h)}. \quad \textit{Ans.}$$

Example 2. (*a*) Write in functional notation: The area A of a square is determined by the length D of its diagonal. (*b*) Find the exact form of this function.

Solution: (*a*) $A = f(D)$. *Ans.*

(*b*) From Figure 28, we find that the area of half the square is the area of a triangle of base D and altitude $\frac{1}{2}D$. Hence,

$$A = f(D) = 2(\tfrac{1}{2})(D)(\tfrac{1}{2}D) = \tfrac{1}{2}D^2. \quad \textit{Ans.}$$

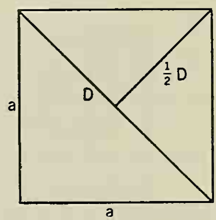


Fig. 28

This same result can also be obtained by using the fact that the diagonal is the hypotenuse of a right triangle whose legs are each a. Hence,

$$D^2 = a^2 + a^2 = 2a^2,$$

or $$a^2 = \tfrac{1}{2}D^2.$$

Since $A = a^2$, we have $A = \frac{1}{2}D^2$, as before.

In this chapter, we have used four different ways to describe a functional relation: (*a*) a verbal statement, (*b*) a table of values, (*c*) a graph, (*d*) a formula. All these ways are useful, and we must

frequently pass from one form of description to another, as has been done in the preceding pages.

On pages 81 to 82, an important property of graphs has been used to find approximately the roots of equations. This property may be easily stated in general: *The abscissas of the points where the graph of the function* y = f(x) *crosses the* X *axis are roots of the equation* f(x) = 0.

We may also deal with functions that involve more than one independent variable. The definitions and notations are similar to those already discussed. For example, $z = f(x, y)$ is read, "z equals a function of x and y." We write $f(a, b)$ to mean that a replaces x and b replaces y in the defining expression of $f(x, y)$.

EXAMPLE 3. If $f(x, y) = \dfrac{x^2 + y^2}{x - y}$, show that $f(tx, ty) = t \cdot f(x, y)$.

Solution: The symbol $f(tx, ty)$ stands for the function obtained by replacing x by tx and y by ty in the given function. Thus,

$$f(tx, ty) = \frac{(tx)^2 + (ty)^2}{(tx) - (ty)} = \frac{t^2x^2 + t^2y^2}{tx - ty}$$

$$= t \cdot \frac{x^2 + y^2}{x - y} = t \cdot f(x, y). \quad \textit{Ans.}$$

EXERCISES 24

1. Express the area A of a circle as a function of its circumference C.

2. Express the surface S of a sphere as a function of the circumference C of a great circle.

3. Express the volume V of a right circular cylinder as a function of its altitude h if the radius is 5 units longer than the altitude.

4. Express the volume V of a cone as a function of its radius r if the altitude h is 6 units shorter than the radius.

5. Express the square of the body diagonal d of a cube as a function of the edge e.

6. Express the volume V of a hemispherical shell as a function of the inner radius r_2 and the outer radius r_1.

7. The lower base b_1 of a trapezoid is three times as long as the upper base b_2, and the altitude h is equal to half of the lower base. Express the area A as a function of the lower base b_1.

8. If $f(x) = x^2 - 9$, find $f(0)$; $f(-1)$; $f(3)$; and $f(5)$.

9. If $h(y) = y^3 + 7y - 6$, find $h(-2)$; $h(3)$; and $h(\frac{1}{3})$.

10. If $F(z) = \dfrac{2z - 8}{z^2 + 9}$, find $F(0)$; $F(4)$; $F(-3)$; and $F(\frac{2}{3})$.

11. If $g(v) = \dfrac{5v + 7}{2v^2 - 1}$, find $g(-2)$; $g(-\frac{7}{5})$; $g(\frac{1}{2})$; and $g(\frac{2}{5})$.

12. Find $f(r)$, $f(-c)$, and $f\left(\dfrac{b}{c}\right)$ if $f(x) = c^2x + cx^2$.

13. Find $h(-k)$, $h(a)$, $h(0)$, and $h\left(\dfrac{a}{c}\right)$ if $h(m) = m^2 - 3am + 2a^2$.

14. If $F(w) = \dfrac{6w}{3w - 2}$, find $F\left(\dfrac{b}{3}\right)$, $F\left(-\dfrac{1}{u}\right)$, and $F\left(\dfrac{4}{x}\right)$.

15. If $g(x) = \dfrac{x + 7}{5x - 8}$, find $g\left(\dfrac{y}{5}\right)$, $g\left(\dfrac{z}{x}\right)$, and $g\left(-\dfrac{2}{b}\right)$.

16. If $F(y) = 2y^2 + 3$, find $\dfrac{F(y + h) - F(y)}{h}$.

17. Find $\dfrac{f(u + h) - f(u)}{h}$ if $f(u) = -\dfrac{1}{2u^2}$.

18. Find $F\left(\dfrac{y - 2}{y}\right)$ if $F(x) = \dfrac{x}{x - 2}$.

19. Find $f\left(\dfrac{x}{x - 3}\right)$ if $f(y) = \dfrac{y}{y - 3}$.

20. If $h(y^2) = 3y^4 - y^2 + 1$, find $h(v^3)$.

21. If $g(x^3) = x^6 - 2x^3 + 4$, find $g(-u^2)$.

22. Find $\dfrac{1}{F(x + 1)} + \dfrac{1}{f(x - 1)}$ if $F(x) = \dfrac{1}{x^2 - 1}$ and $f(x) = \dfrac{1}{x^2 + 1}$.

23. Find $\dfrac{1}{g(2/x)} - \dfrac{1}{h(2/x)}$ if $g(x) = \dfrac{cx + b}{c}$ and $h(x) = \dfrac{cx - b}{c}$.

24. If $f(x, y) = 2x^2 + 4xy + y^2$, find $f(x, -y)$, $f(-x, y)$, and $f(-x, -y)$.

25. If $h(x, y) = x^2 - 3y^2 + 15$, find $h(-x, y)$, $h(x, -y)$, and $h(-x, -y)$.

26. Show that $f(mx, my) = m^2f(x, y)$ if $f(x, y) = 3x^2 + 4xy - 5y^2$.

27. Show that $F(tx, ty) = t^3F(x, y)$ if $F(x, y) = x^3 - 4x^2y + y^3$.

28. If $G(u, v) = 4u^2 - 4v^2$, find $G(r + s, r - s)$.

29. If $f(x, y) = 3x^2 + 3y^2$, find $f(u + v, u - v)$.

Chapter 6

LINEAR EQUATIONS IN MORE THAN ONE VARIABLE

32. Linear Equations in Two Variables

In the preceding chapter, we noted that a linear equation in two variables, such as

$$ax + by + c = 0$$

has a straight line for its graph. It will be of interest to discuss the geometrical situation that arises when the graphs of two linear equations are drawn on the same set of axes.

33. Graphical Solution of Linear Equations

The lines whose equations are

$$2x - 3y = 3$$

and $$x + y = 4,$$

respectively, are shown in Figure 29. The two lines intersect at the point (3, 1). For, if $x = 3$ and $y = 1$,

$$2x - 3y = 6 - 3 = 3,$$

and $$x + y = 3 + 1 = 4;$$

that is, the number pair (3, 1), upon substitution, reduces both equations simultaneously to numerical identities.

This idea leads to the definition: *A number pair* (x_1, y_1) *that satisfies simultaneously a system of two linear equations is a* **solution of the system.**

For instance, (3, 1) is a solution of the system of equations

$$2x - 3y = 3$$

and

$$x + y = 4.$$

Since two straight lines cannot intersect at more than one point without being coincident, we may speak of (3, 1) as *the* solution.

Any two distinct lines that are not parallel will intersect in exactly one point. The coordinates of this point will then constitute the solution of the set of two equations that describe the lines.

If the set of equations describes two parallel lines, there will be no point of intersection and the equations will have no simultaneous solution. We sometimes say that such equations are **inconsistent**.

Illustration: The two equations

$$2x + 3y = 6$$

and

$$2x + 3y = 12$$

are inconsistent equations; for, no number pair can make the expression $2x + 3y$ have the values 6 and 12 simultaneously. The graphs of these two parallel lines are shown in Figure 30.

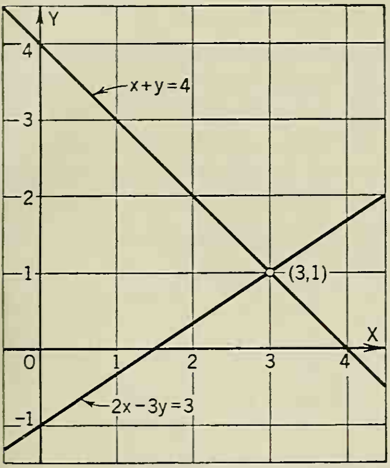


Fig. 29

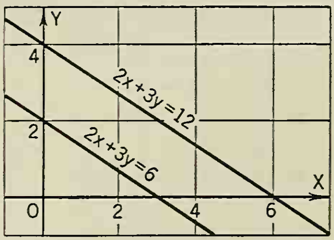


Fig. 30

In general, inconsistent equations may be detected by inspection; for such equations may always be written in the form

$$ax + by = c_1$$

and

$$ax + by = c_2,$$

where the left members are identical and the right members are different. If the left members of inconsistent equations are not identical, they may be made so by multiplying or dividing both sides of one of the equations

by a suitable nonzero constant. For example,

$$5x - 2y = 17$$

and

$$15x - 6y = 10$$

are inconsistent equations. The second equation, when both sides are divided by 3, becomes

$$5x - 2y = \tfrac{10}{3}.$$

However, if two linear equations can be made completely identical by such a multiplication or division, the equations are consistent and represent the same line. Equations of this kind are called **dependent.**

Illustration: The equations

$$5x - 2y = 6$$

and

$$15x - 6y = 18$$

are dependent equations; for, if the second equation is divided through by 3, it becomes identical with the first; hence, any solution of one of these equations is a solution of the other.

EXERCISES 25

Solve graphically each of the following sets of equations:

1. $2x + 3y = 12$
$3x + 2y = 13$

2. $x + 5y = -1$
$2x + 7y = 1$

3. $3y - 4x = 23$
$5y + 9x = 7$

4. $6x - 5y = 4$
$8x - 7y = 6$

5. $10 - 2x - 5y = 0$
$15 - 3x + 8y = 0$

6. $12 + 7x - 3y = 0$
$8 + 5x - 2y = 0$

7. $2x + y = 7$
$2x + y = 10$

8. $3x - 4y = 12$
$6x - 8y = 15$

9. $5x - 6y = 6$
$3x + 10y = 7$

10. $11x + 7y = -45$
$5x - 9y = 10$

11. $\frac{x}{3} - \frac{y}{4} = -\frac{1}{2}$
$\frac{x}{2} - \frac{y}{6} = \frac{1}{2}$

12. $\frac{3x}{8} + \frac{2y}{15} = \frac{5}{6}$
$\frac{x}{2} - \frac{7y}{20} = \frac{15}{4}$

13. $7x + 3y + 9 = 0$
$16x - 20y - 13 = 0$

14. $3x - 4y - 39 = 0$
$2x - y - 13 = 0$

Show that each of the following systems represents either a pair of parallel lines or a pair of coincident lines.

15. $3x - 4y = 8$
$12y - 9x = -24$

16. $3x + 5y = 15$
$6x + 10y = 45$

17. $4x - 7y = 28$
$8x - 14y = 42$

18. $3x - 2y = 4$
$\frac{x}{2} - \frac{y}{3} = \frac{2}{3}$

19. $\frac{x}{9} + \frac{y}{15} = \frac{1}{3}$
$\frac{x}{3} + \frac{y}{5} = 1$

20. $\frac{x}{2} - \frac{2y}{3} = 0$
$\frac{3x}{8} - \frac{y}{2} = 2$

34. Analytic Solution of Systems of Linear Equations

Usually, the solution of a pair of linear equations in two variables can be read only approximately from the graph, whereas an exact solution may be required. Such an exact solution can always be obtained by properly combining the equations whenever they are not inconsistent or dependent.

By referring to the graphical representation, we see that the particular number pair (x, y) that describes the point of intersection of the two lines is the same for both equations. Hence, if each equation were solved for y in terms of x, we would have two expressions for the ordinate of the point of intersection in terms of the abscissa. By equating these expressions, we would obtain a single equation in one unknown. The root of this equation is the required value of x. The corresponding value of y can be found by substituting the value of x into one of the expressions for y.

Example 1. Solve the set of equations

$$2x - 3y = 3,$$

and

$$x + y = 4.$$

Solution: Solve each of the equations for y to obtain

$$y = \frac{2x - 3}{3}, \tag{1}$$

and

$$y = 4 - x. \tag{2}$$

Equate these two expressions for y, and solve the resulting equation. We have

$$\frac{2x - 3}{3} = 4 - x;$$

$$2x - 3 = 12 - 3x;$$

$$x = 3.$$

Substitute 3 for x in either Equation (1) or (2) to find $y = 1$.

The required solution, which has already been verified on page 90, is (3, 1). *Ans.*

The method used in Example 1 is the *method of comparison.* Other methods differ from this one only in the manner of combining the equations in order to obtain a single equation in one unknown.

The *method of addition or subtraction* consists in multiplying one or both of the equations by constants chosen so that we obtain two equations in which one of the unknowns, say y, has the same numerical coefficient. The corresponding members of these two equations are added or subtracted (whichever is necessary) to eliminate y. The resulting equation is solved for x. A similar procedure may be used to find the value of y. The proposed solution should always be checked in the given equations.

EXAMPLE 2. Solve the equations

$$2x - 3y = 3, \tag{1}$$

$$x + y = 4. \tag{2}$$

Solution: Multiply the members of Equation (2) by 3 to obtain

$$3x + 3y = 12, \tag{3}$$

an equation in which the coefficient of y is the same except for sign as in the first of the given equations. Now, add the corresponding members of Equations (1) and (3).

$$\begin{aligned} 2x - 3y &= 3, & (1)\\ \underline{3x + 3y} &= \underline{12,} & (3)\\ 5x \qquad &= 15. \end{aligned}$$

Thus, we find $$x = 3.$$

In a similar way, multiply both members of Equation (2) by 2 and subtract Equation (1), member by member, from the result.

$$\begin{aligned} 2x + 2y &= 8, & (4)\\ \underline{2x - 3y} &= \underline{3,} & (1)\\ 5y &= 5. \end{aligned}$$

Therefore, $$y = 1.$$

We see that the solution is the same as that obtained in Example 1, namely, (3, 1). *Ans.*

NOTE: It is not necessary to find both unknowns by the method of

addition or subtraction. For instance, in Example 2, the value of y can be obtained more easily by substituting 3 for x in Equation (2).

In the *method of substitution*, one of the equations is solved for one of the unknowns, and the result is substituted into the other equation. This procedure yields a single equation in one unknown. The remaining steps are as before.

EXAMPLE 3. Solve the equations

$$3x + 4y = 24, \tag{1}$$

$$2x + y = 11. \tag{2}$$

Solution: We solve Equation (2) for y in terms of x and substitute the resulting expression in place of y in Equation (1). Thus,

$$y = 11 - 2x, \tag{3}$$

and
$$3x + 4(11 - 2x) = 24,$$
that is,
$$-5x + 44 = 24,$$
or
$$-5x = -20,$$
and
$$x = 4.$$

If we substitute 4 for x in Equation (3), we get

$$y = 11 - 8 = 3.$$

If, as a check, (4, 3) is substituted into the left member of Equation (1), the result is

$$3x + 4y = 12 + 12 = 24;$$

hence, the required solution is (4, 3). *Ans.*

It often happens that systems of equations that are not in standard linear form ($ax + by = c$) can be put into this form and, hence, can be solved by the preceding methods. The next example illustrates this idea.

EXAMPLE 4. Solve the equations

$$\frac{2}{x - y} + \frac{3}{x + y} = 3,$$

$$\frac{8}{x - y} - \frac{9}{x + y} = \frac{3}{2}.$$

Solution: The fact that x and y occur in these equations only in the

combinations $\dfrac{1}{x-y}$ and $\dfrac{1}{x+y}$ suggests that we replace each of these fractions by a new letter. Accordingly, let

$$\frac{1}{x-y} = u, \tag{1}$$

and

$$\frac{1}{x+y} = v. \tag{2}$$

Then, the given equations become

$$2u + 3v = 3,$$

and

$$8u - 9v = \tfrac{3}{2}.$$

The last two equations may be solved to find $u = \frac{3}{4}$ and $v = \frac{1}{2}$. Hence, by reference to (1) and (2), we have

$$\frac{1}{x-y} = \frac{3}{4} \quad \text{or} \quad x - y = \frac{4}{3}, \tag{3}$$

and

$$\frac{1}{x+y} = \frac{1}{2} \quad \text{or} \quad x + y = 2. \tag{4}$$

Equations (3) and (4) may now be combined to obtain the required solution $(\frac{5}{3}, \frac{1}{3})$. *Ans.*

Check: If $x = \frac{5}{3}$ and $y = \frac{1}{3}$

First equation: Left Member	Right Member
$= \dfrac{2}{\frac{5}{3} - \frac{1}{3}} + \dfrac{3}{\frac{5}{3} + \frac{1}{3}}$	$= 3$
$= \frac{3}{2} + \frac{3}{2}$	
$= 3$	

Second equation: Left Member	Right Member
$= \dfrac{8}{\frac{5}{3} - \frac{1}{3}} - \dfrac{9}{\frac{5}{3} + \frac{1}{3}}$	$= \frac{3}{2}$
$= 6 - \frac{9}{2}$	
$= \frac{3}{2}$	

EXERCISES 26

Solve each of the systems of equations in Problems 1 to 6 by three methods: (*a*) comparison; (*b*) addition or subtraction; (*c*) substitution.

1. $x + y = 4$
$x - 4y = 19$

2. $2x + 3y = -2$
$3x - 2y = -16$

3. $3x + y = 10$
$4x + 5y = -16$

4. $7x - 4y = -1$
$6x - 7y = -33$

5. $9x + 2y - 37 = 0$
$5x + 6y - 45 = 0$

6. $8x + 7y + 36 = 0$
$10x - 9y - 26 = 0$

In Problems 7 to 28 solve for x and y unless otherwise indicated to the right of the second equation. Use any convenient method.

7. $3x + 5y = 11$
$8x - 12y = -15$

8. $6x + 2y + 1 = 0$
$9x + 4y + 4 = 0$

9. $\frac{x}{5} + \frac{y}{4} = 3$
$\frac{x}{10} - \frac{y}{2} = -1$

10. $\frac{r}{3} + \frac{s}{5} = 5$
$\frac{2r}{15} + \frac{4s}{15} = 2;\quad r, s$

11. $\frac{7u + 5v - 2}{u + 3v} = -8$
$\frac{6u - 3v + 10}{5u + 6v} = 5;\quad u, v$

12. $\frac{9y - 8x + 14}{3x - 4y - 4} = -5$
$\frac{8y - 7x + 15}{2y + 5x - 3} = -\frac{1}{2}$

13. $\frac{2y - 5x}{3} + 5 = \frac{y + 27}{4} - 2x$
$\frac{x + 1}{3} + y = \frac{6y - 5x}{7}$

14. $\frac{k - 3}{6} - \frac{m + 10}{7} = \frac{8 - 3m}{4}$
$\frac{2m - 3}{5} + \frac{k - 4}{5} = \frac{k - 5}{2};\quad k, m$

15. $(x + 2)(y - 4) = xy$
$(x + 8)(y - 10) = xy$

16. $(3a + 2)(2b - 3) = 6ab$
$(4a + 5)(b - 5) = 4ab;\quad a, b$

17. $3x + 4y = 7c + 6d$
$2x - y = c - 7d$

18. $7c + 6d = 3x + 4y$
$c - 7d = 2x - y;\quad c, d$

19. $ax + by + a^2 + b^2 = 0$
$bx + ay + 2ab = 0$

20. $by + (a + b)x = 2ab$
$ay + (a - b)x = a^2 - b^2$

21. $a_1x + b_1y = c_1$
$a_2x + b_2y = c_2$

22. $ex + fy = g$
$hx - ky = m$

23. $\frac{4}{x} + \frac{12}{y} = 5$
$\frac{5}{x} - \frac{6}{y} = 1$

24. $\frac{5}{x} - \frac{7}{y} = 18$
$\frac{3}{x} + \frac{11}{y} = -12$

25. $\dfrac{2}{3x} + \dfrac{3}{4y} = \dfrac{7}{2}$
$\dfrac{1}{3x} - \dfrac{1}{2y} = 0$

26. $\dfrac{5}{9x} - \dfrac{1}{3y} = -3$
$\dfrac{1}{3x} - \dfrac{3}{8y} = -\dfrac{5}{2}$

27. $\dfrac{2}{x-y} + \dfrac{1}{x+y} = \dfrac{2}{3}$
$\dfrac{5}{x-y} - \dfrac{6}{x+y} = \dfrac{1}{4}$

28. $\dfrac{1}{3x-y} + \dfrac{3}{x+3y} = 2$
$\dfrac{10}{3x-y} - \dfrac{9}{x+3y} = 7$

In each of the following four problems, show that the system of three equations has a solution. (Hint: Show that the solution of two of the equations satisfies the third equation.)

29. $x - 5y = 27$
$2x + 3y = 2$
$5x + 8y = 3$

30. $9x + 4y = 10$
$5x - 2y = 6$
$3x + 14y = 2$

31. $(a-b)x - (a+b)y = 2a^2 - 2b^2$
$(a+b)x + (a-b)y = 4ab$
$x + y - 2b = 0$

32. $cx + dy = c^2 + d^2$
$cx - dy = c^2 - 2cd - d^2$
$x + 2y = 3c + d$

35. Linear Equations in More than Two Variables

The procedure for finding the solution of any number of linear equations in the same number of variables is an immediate extension of the procedure for handling a system of two equations in two unknowns. For the sake of definiteness, let us consider three equations in three unknowns. The equations may be combined in two sets, eliminating the same unknown in both combinations. The result is a set of two equations in two unknowns, for which the methods of solution have already been discussed.

Example 1. Solve the equations

$$x + y + 2z = 7, \tag{1}$$

$$x - y - 3z = -6, \tag{2}$$

$$2x + 3y + z = 4. \tag{3}$$

Solution: An inspection of the system shows that Equations (1) and (2) may be added to eliminate y. Furthermore, if Equation (2) is multiplied by 3 and the result is added to Equation (3), a second equation is obtained with y missing. These two new equations are

$$2x - z = 1 \tag{4}$$

and

$$5x - 8z = -14. \tag{5}$$

Any of the methods previously studied gives $x = 2$, $z = 3$ as the solution of the system (4) and (5). If we substitute $x = 2$ and $z = 3$ into Equation (1), we find $y = -1$. Hence, the required solution, which the student should check, is $(2, -1, 3)$. *Ans.*

EXAMPLE 2. Solve the equations

$$\begin{aligned} r + s - t &= 1, \\ s + t - u &= 2, \\ t + u - r &= 3, \\ u + r - s &= 4. \end{aligned}$$

Solution: In order to see more clearly what steps should be taken to perform the elimination, we rewrite the equations so that the letters of the left member of each are in alphabetical order.

$$r + s - t \qquad = 1, \tag{1}$$

$$s + t - u = 2, \tag{2}$$

$$-r \qquad + t + u = 3, \tag{3}$$

$$r - s \qquad + u = 4. \tag{4}$$

Inspection now shows that Equations (1) and (2) may be added to eliminate t; and (1) and (3) may also be added to eliminate t. These two results along with (4) will give us a system of three equations in three unknowns.

Hence, adding (1) and (2) $\quad r + 2s - u = 3, \qquad (5)$

and adding (1) and (3) $\quad s + u = 4. \qquad (6)$

We also have $\quad r - s + u = 4. \qquad (4)$

Equations (4), (5), and (6) are three equations in r, s, and u.

Since r is missing from Equation (6), we seek another combination which will eliminate r. This combination is found by subtracting (4) from (5) with the result

$$3s - 2u = -1. \tag{7}$$

Equations (6) and (7) are two equations in the two unknowns s and u. Hence, our previous methods may be used to complete the solution. We find $s = \frac{7}{5}$ and $u = \frac{13}{5}$. If these two values are substituted into Equation (4), there is obtained $r = \frac{14}{5}$. Finally, $t = \frac{16}{5}$ may be found

by substituting the values of r and s into Equation (1). Hence, the solution is

$$r = \tfrac{14}{5},\ s = \tfrac{7}{5},\ t = \tfrac{16}{5},\ u = \tfrac{13}{5}. \quad \textit{Ans.}$$

The check is left to the student.

EXERCISES 27

Solve each of the following systems of equations:

1. $2x + y - 3z = 11$
$x - 2y + 4z = -3$
$3x + y - 2z = 12$

2. $5x + 3y - 2z = 1$
$2x + 4y + z = 0$
$4x - 7y - 7z = 6$

3. $3x - 2y = 3$
$5y - 7z = 4$
$8z - 5x = 19$

4. $x + 3y = 19$
$y + 3z = 10$
$z + 3x = -5$

5. $8r + 3s - 18t = 1$
$16r + 6s - 6t = 7$
$4r + 9s + 12t = 9$

6. $u + 3v - 2w = 10$
$7u - 5v + 4w = -4$
$2u + v + 3w = 4$

7. $cx + by = c$
$by - az = b$
$az - cx = a$

8. $ax - by - cz = 1$
$bx - cy - az = 1$
$cx - ay - bz = 1$

9. $\frac{2}{x} + \frac{3}{y} + \frac{1}{z} = 4$
$\frac{3}{x} - \frac{5}{y} + \frac{2}{z} = -5$
$\frac{4}{x} - \frac{6}{y} + \frac{3}{z} = -7$

10. $\frac{1}{x} + \frac{6}{y} + \frac{1}{z} = 6$
$\frac{2}{x} + \frac{3}{y} - \frac{2}{z} = 8$
$\frac{2}{x} + \frac{4}{z} = 3$

11. $x - y - 2z = 7$
$2y + 3z + w = -9$
$3z + 2w + x = -11$
$w + x + 4y = 4$

12. $4r + s + t = 10$
$s - t - 2u = 14$
$2t + u - 3r = 4$
$u + 3r + 2s = 0$

13. $2x + y - 2z - w = -8$
$3x - 2y - 3z + 4w = 6$
$x + y + 2z + 2w = 9$
$4x - 3y + z - 5w = 3$

14. $x + 2y + z + w = 0$
$3x + y + 2z - 2w = 7$
$x - 5y - 3z + 2w = 7$
$2x + 4y + 2z - w = 3$

15. $\frac{1}{x} + \frac{1}{y} - \frac{1}{z} = 9$
$\frac{1}{y} - \frac{1}{z} + \frac{1}{w} = 12$
$\frac{1}{w} + \frac{1}{x} - \frac{1}{z} = 13$
$\frac{1}{x} + \frac{1}{y} + \frac{1}{w} = 11$

16. $\frac{1}{x} + \frac{3}{y} - \frac{2}{z} = 5$
$\frac{2}{y} + \frac{1}{z} + \frac{4}{w} = -3$
$\frac{1}{z} + \frac{2}{w} - \frac{1}{x} = -3$
$\frac{2}{w} + \frac{1}{x} - \frac{2}{y} = 8$

36. Statement Problems Involving more than One Unknown

Many problems lend themselves naturally to the use of two or more unknowns in their solution. This will be illustrated by examples.

Example 1. An airplane makes a 700-mile trip against a head wind in 2 hr 20 min. The return trip takes 2 hr, the wind now being a tail wind. If the plane maintains a constant speed with respect to still air, and the speed of the wind does not vary, find the still-air speed of the plane and the speed of the wind.

Solution: Let x mph = the still-air speed of the plane,

and y mph = the speed of the wind.

Then, $(x + y)$ mph = the ground speed of the plane with the tail wind,

and $(x - y)$ mph = the ground speed against the head wind.

Thus, for the trip against the wind,

$$x - y = \frac{\text{distance}}{\text{time}} = \frac{700}{\frac{7}{3}} = 300; \tag{1}$$

and for the trip with the wind,

$$x + y = \frac{\text{distance}}{\text{time}} = \frac{700}{2} = 350. \tag{2}$$

Now, Equations (1) and (2) constitute a system of two equations in two unknowns. The student may solve this system to find

$$x = 325 \quad \text{and} \quad y = 25. \quad \textit{Ans.}$$

Check: For these results, the ground speed on the trip against the wind would be 300 mph; the time taken would be $\frac{700}{300}$ = 2 hr 20 min. On the return trip, the ground speed would be 350 mph; the time taken would be $\frac{700}{350}$ = 2 hr. Therefore, the plane's still-air speed is 325 mph, and the wind speed is 25 mph.

Example 2. A 100-gal vat in a chemical plant has two intake pipes, a large one carrying water and a small one carrying acid. If both pipes are turned on together, the vat can be filled in 40 min. Ordinarily, however, the water pipe is allowed to run for 45 min and is shut off

before the acid is turned on. It then takes the acid 20 min to fill the tank. What is the ordinary strength of the solution?

Solution: Let w gal per min = the delivery rate of the water pipe,

and a gal per min = the rate of the acid pipe.

Then $40w$ = the number of gallons of water delivered in 40 min,

and $40a$ = the number of gallons of acid delivered in 40 min.

Thus, $$40w + 40a = 100,$$

or $$2w + 2a = 5. \tag{1}$$

Similarly, $45w$ = the number of gallons of water delivered in 45 min,

and $20a$ = the number of gallons of acid delivered in 20 min.

Hence, $$45w + 20a = 100,$$

or $$9w + 4a = 20. \tag{2}$$

The student may solve (1) and (2) simultaneously to find

$$w = 2 \quad \text{and} \quad a = \tfrac{1}{2}.$$

Therefore, the rate of flow of acid is $\frac{1}{2}$ gal per min. In 20 min there would be 10 gal of acid delivered, and the solution would be 10 per cent acid. *Ans.*

Check:

$$\begin{aligned} (40)(2) &= 80 && \text{(gal of water delivered in 40 min)} \\ (40)(\tfrac{1}{2}) &= \underline{20} && \text{(gal of acid delivered in 40 min)} \\ \text{Sum} &= 100 \end{aligned}$$

$$\begin{aligned} (45)(2) &= 90 && \text{(gal of water delivered in 45 min)} \\ (20)(\tfrac{1}{2}) &= \underline{10} && \text{(gal of acid delivered in 20 min)} \\ \text{Sum} &= 100 \end{aligned}$$

EXERCISES 28

1. If the numerator of a certain fraction is doubled and the denominator is increased by 8, the value of the fraction becomes $\frac{1}{4}$. If the numerator is increased by 7 and the denominator is tripled, the value becomes $\frac{5}{24}$. Find the fraction.

2. If 7 is added to both the numerator and denominator of a fraction, the value of the fraction is $\frac{9}{10}$. If 3 is subtracted from both numerator and denominator, the value becomes $\frac{4}{5}$. Find the fraction.

3. Longs Peak is 145 ft higher than Pikes Peak, and the sum of their elevations is 776 ft less than the elevation of Mt. Everest, which is 29,141 ft. Find the elevation of each of the lower peaks.

4. Two bank accounts together total $10,000. If $750 is taken from the first account and put into the second, each will have the same amount. How much is there in each account?

5. A man has part of $30,000 loaned at 2 per cent interest and the rest at $2\frac{1}{2}$ per cent interest. If the income from the money loaned at 2 per cent is $60 more than that loaned at $2\frac{1}{2}$ per cent, what is the amount of each loan?

6. A grocer bought a number of cases of vegetables for $147. Some of the vegetables were paid for at the rate of 3 cases for $7 and the rest at the rate of 4 cases for $7. When the grocer sold all the vegetables for $3 per case, he cleared $78. How many cases of each kind did he buy?

7. A man bought eighteen pairs of skis and sixteen pairs of cable bindings for $452. He found that he had underordered, and he then bought twelve pairs of skis and eight pairs of cable bindings for $292. Find the prices of the skis and bindings.

8. The members of an outing club bought a plane. If there had been six members more, each would have paid $25 less; and if there had been four members fewer, each would have paid $25 more. Find the price of the plane and the number of members in the club.

9. The boiling point of water at sea level is 212°F (Fahrenheit) or 100°C (Centigrade), and the freezing point is 32°F or 0°C. If the equation involving F and C is of the form $F = aC + b$, find a and b.

10. A weight of 40 lb is placed on one side of a lever so that it balances a 60-lb weight on the other side. If the 60-lb weight is increased to 80 lb, the 40-lb weight must be moved 2 ft farther away from the fulcrum to maintain the balance. Find the original distance between the 40-lb weight and the 60-lb weight.

11. The fulcrum of a lever is so placed that weights of 180 lb and 240 lb put at the ends are in balance. When 60 lb is added to the 240 lb weight, the fulcrum must be moved 9 inches closer to the 240-lb weight to preserve the balance. Find the length of the lever if its weight is negligible.

12. A motor boat can run 55 miles downstream in 2 hr 45 min and can return in 4 hr 35 min. What is the speed of the boat in still water and the speed of the current?

13. A plane travels 1050 miles with a tail wind in 2 hr 30 min and returns in 3 hr 30 min against the head wind. Find the speed of the plane in still air and the speed of the wind.

14. A man rows 21 miles up a river and back in 6 hr 30 min. He finds that he can row 6 miles up the river in the same time that he can row 7 miles down the river. Find the speed of the boat in still water and the speed of the current.

15. An aviator flies 1680 miles against a wind and returns with the wind. His round trip requires $7\frac{1}{2}$ hr. He finds that he can fly 40 miles with the wind in the same time that he can fly 35 miles against the wind. Find the speed of the wind and the speed of the plane in still air.

16. On a cross-country flight, it took aviator A 18 min longer than aviator B to fly 450 miles. If A had doubled his speed, he would have taken 36 min less time than B. Find A's and B's speeds on the flight.

17. A man drives from his office to his home in the country. When he drives at 50 mph he arrives 4 min earlier than usual; and when he drives at 40 mph, he arrives 5 min later than usual. How far is his home from the office, and what is the usual time it takes him to drive the distance?

18. A 90 per cent acid solution is diluted with water to make a solution 60 per cent acid. When 2 gal of water is added to dilute it again, the solution becomes 40 per cent acid. How much water was added to the solution the first time?

19. In a pickling factory a saturated solution of salt is called a 100 per cent solution. A 30 per cent solution was made by adding water to a 40 per cent solution. Then the 30 per cent solution was diluted to a 20 per cent solution by adding 3 gal of water. How much water was first added to the 40 per cent solution? How much of the 40 per cent solution was there initially?

20. On counting his cash, a grocer finds that he has \$48 consisting of 294 pieces of silver in half dollars, quarters, and dimes. There are $3\frac{1}{2}$ times as many dimes as quarters. How many coins of each denomination are there?

21. A sum of money is composed of quarters and dimes, and there are four times as many dimes as quarters. If there were fifty-four more dimes and twenty more quarters, the total value would be doubled. Find the sum and the number of quarters and dimes.

22. Three machines, A, B, and C, operating together can do a job in 8 hr. If A operates for 2 hr and B for 3 hr, one fourth of the job can be done; and if B operates for 5 hr and C for 14 hr, one half of the job can be done. How many hours would it take each machine operating alone to do the job?

23. Three contractors, A, B, C, can do a construction job in 20 days. A and B together can do the work in 30 days, and B and C can do the work in 40 days. How many days would it take each contractor alone to do the work?

24. Three pipes supply a water tank. The tank can be filled by pipes A and B running for 4 hr, or by pipes B and C running for 6 hr, or by pipes C and A running for 8 hr. What length of time is required for each pipe running alone to fill the tank?

25. A reservoir can be filled by a supply pipe and drained by two other pipes. When all the valves are open, the reservoir can be filled in 36 hr. When the valve on the supply pipe is closed, the other pipes can drain the reservoir in 60 hr. With the supply pipe and one of the drain pipes running, the reservoir can be filled in 30 hr. Find the time required for each drain pipe to empty the reservoir with the remaining two valves closed.

26. The points $(0, 4)$, $(-12, -2)$, and $(-2, 6)$ lie on a curve whose equa-

tion is of the form $y = ax^2 + bx + c$. Find a, b, and c. (HINT: The coordinates of each point must satisfy the equation.)

27. The points given in the preceding problem lie on a curve whose equation is of the form $x = ay^2 + by + c$. Find a, b, and c.

28. It is shown in analytic geometry that the equation of a circle is of the form $x^2 + y^2 + Dx + Ey + F = 0$. If a circle passes through the three points given in Problem 26, find its equation.

Chapter 7

EXPONENTS AND RADICALS

37. Rational and Irrational Numbers

When we discussed the correspondence between the numbers of algebra and the points on our number scale, we noted that there is a point on the scale which corresponds to each of the following numbers: The positive and negative integers and zero, $\cdots -3, -2, -1, 0, 1, 2, 3, \cdots$, and the fractions that may be formed by writing the indicated quotient of two integers, such as $\frac{1}{2}$, $-\frac{1}{3}$, $\frac{22}{7}$, and so on. The decimal fractions are also included because 0.231, for example, is only another way of writing $\frac{231}{1000}$.

Any number that can be written as the quotient of two integers is called a ***rational number.*** (The student should think of the word *ratio* which is the first part of *ratio*nal.) Thus, all the numbers described in the preceding paragraph are rational, and the operations that we have considered up to this point have all dealt with rational numbers.

Any number that is not rational is called **irrational.** We shall consider next an operation for which we frequently need irrational numbers. This operation is *the extraction of roots,* the inverse of the process of raising a number to a power.

38. Roots

If n *is a positive integer, we define an* n*th root of a number* **a** *to be any number* x *such that*

$$x^n = a.$$

Illustrations: (*a*) A square (second) root of 4 is 2 because $2^2 = 4$. Another square root of 4 is -2 because $(-2)^2 = 4$.

(*b*) A cube (third) root of 27 is 3 because $3^3 = 27$.

(*c*) A cube root of -64 is -4 because $(-4)^3 = -64$.

(*d*) A fourth root of $\frac{16}{81}$ is $\frac{2}{3}$ because $(\frac{2}{3})^4 = \frac{16}{81}$.

Not all rational numbers have rational roots. For example, the number 2 has no rational number for its square root. This may easily be shown as follows. Let us assume that 2 does have a rational square root. Such a root could be written in the form $\frac{a}{b}$, where a and b are *integers with no common factor*. From the definition of a square root, it follows that

$$\left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2} = 2$$

and $$a^2 = 2b^2.$$

Since a and b are integers, a^2 and b^2 are also integers. Furthermore, $2b^2$ is an even integer, from which we conclude that a^2 is an even integer. Since only even integers have even integers as their squares,

$$a = 2c, \quad \text{where } c \text{ is an integer,}$$

and $$a^2 = 4c^2.$$

Since $2b^2$ also equals a^2, it follows that

$$4c^2 = 2b^2.$$

Next we divide each member by 2 to get

$$2c^2 = b^2,$$

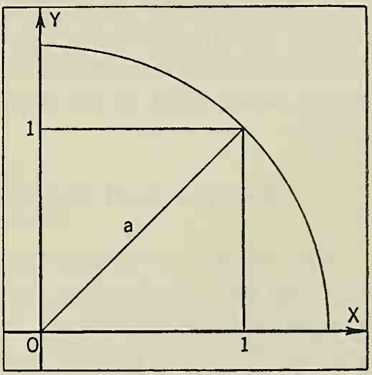


Fig. 31

from which it must follow in the same way that b is an even integer. Hence,

$$b = 2d, \quad \text{where } d \text{ is an integer.}$$

However, if $a = 2c$, and $b = 2d$, a and b have the common factor 2 which contradicts our assumption that a and b have no common factor! Hence, 2 has no rational square root.

In order to avoid the conclusion that there is no number whose square is 2, consider the construction in Figure 31. A square whose side

is one unit long is drawn with a right angle at the origin and two sides along the axes. The length a of the diagonal is marked off by means of compasses on the X axis. Since the square of the hypotenuse of a right triangle equals the sum of the squares of the two sides, we have

$$a^2 = 1^2 + 1^2 = 2.$$

Consequently, if we wish to assign a number for each point on the number scale, the point where the circle intersects the X axis must correspond to a number whose square is 2.

It can be shown in a manner similar to that used above that no integer which is not an exact nth power of another integer has a rational nth root. For example, among the integers from 2 to 10, only three have rational roots, namely,

4 with the rational square roots 2 and -2;

9 with the rational square roots 3 and -3;

8 with the rational cube root 2.

Another example of an irrational number is $\pi = 3.14159 \cdots$, the ratio of the circumference of a circle to its diameter. (The three dots indicate that this is an endless decimal.) Many more irrational numbers will be encountered by the student who goes further into mathematical work.

39. The Real Number System

The system of numbers which consists of the rational and the irrational numbers is called the **real number system.** We take it as a fundamental axiom that to each point on the number scale there corresponds exactly one real number, and conversely. The real number system as just defined has the following important properties:

(1) *The sum, difference, product, and quotient (division by zero excepted) of any two real numbers is a real number.*

(2) *Every positive integral power of a real number is a real number.*

(3) *If* n *is an odd positive integer, every real number has one real* n*th root.*

(4) *If* n *is an even positive integer, every positive real number has two real* n*th roots; these roots are equal numerically but opposite in sign.*

(5) *If* n *is an even positive integer, a negative real number has no real* n*th root.* (Since an even power of any positive or negative real number is a positive number.)

40. Radicals

We shall confine our next discussion entirely to real numbers. The name **principal *n*th root** is used to designate the one real root when n is odd and the one positive root when n is even. We use the expression $\sqrt[n]{a}$ to stand for the principal nth root of a. The entire expression is called a **radical expression,** or, simply, a **radical;** the symbol $\sqrt{\ }$ is a **radical sign;** the n is called the **index,** or **order,** of the radical; and a is called the **radicand.** The horizontal bar usually associated with the radical sign is a grouping symbol customarily drawn above the radicand. In the case of a square root, the index 2 is usually understood but not written.

Illustrations: (a) $\sqrt{a+b}$ stands for the positive square root of the quantity $a + b$.

(b) $\sqrt[3]{-27} = -3$.

(c) $\sqrt[4]{16} = 2$. (NOT ± 2.)

It is convenient here to define the **absolute** or **numerical value** of a signed number n to be n if n is positive and to be $-n$ if n is negative. The absolute value of a number is indicated by placing the number between two vertical bars. Thus,

$$|+2| = 2;$$

$$|-2| = 2;$$

$$|-1.76| = 1.76;$$

and

$$\sqrt{x^2} = |x|.$$

It is important to understand that for roots of even order the use of the radical sign demands the use of the positive root. It is correct to say that the square roots of x^2 are x and $-x$; in symbols, we write $\sqrt{x^2} = |x|$ and $-\sqrt{x^2} = -|x|$. Thus, if $a > 3$, $\sqrt{(a-3)^2} = a - 3$; but if $a < 3$, $\sqrt{(a-3)^2} = 3 - a$.

EXERCISES 29

1. Give five examples of (a) rational numbers; (b) irrational numbers.

2. Write each of the following rational numbers as a common fraction in its lowest terms: $1\frac{1}{3}$; $2\frac{1}{2}$; 0.12; 0.036; the endless repeating decimal, $0.333 \cdots$; the endless repeating decimal, $0.111 \cdots$.

3. Classify each of the following numbers as rational or irrational: $\sqrt{3}$; 1.7; 3.27; $\sqrt[3]{12}$; 0.33; π; $\sqrt[4]{16}$; $\sqrt[3]{64}$; 0.0011; $2 + \sqrt{5}$.

4. Find all the rational nth roots of the integers from 2 to 100.

5. Calculate the altitude of the equilateral triangle each of whose sides is 4; 7; $6\sqrt{3}$; $\sqrt{5}$. Leave your answers in radical form, unless the radicand is a perfect square. Is the length of the altitude of an equilateral triangle always an irrational number?

6. Is the length of the diagonal of a square always an irrational number? Is it possible for the side and the diagonal to be rational at the same time?

41. Tables of Square and Cube Roots

The most practical way to obtain square roots and cube roots of numbers is to read them from a table. Table I (see Appendix) is a table giving square roots and cube roots of numbers from 1.0 to 10.0 at intervals of 0.1. More extensive tables are available in standard collections of mathematical tables. It should be understood that most of the entries in such a table are only rational approximations correct to the number of decimal places given. Both in the construction and in the use of such tables, we employ the process of **rounding off,** that is, of writing, with a specified number of decimal places, the closest possible approximation to a given number. We shall use the following rules whenever rounding off takes place:

(1) *If the part to be dropped amounts to less than 5 units in the first discarded place, drop it.*

(2) *If the part to be dropped amounts to more than 5 units in the first discarded place, increase the last retained digit by one unit.*

(3) *If the part to be dropped amounts to exactly 5 units in the first discarded place, follow whichever of the above procedures makes the last retained digit even.*

Illustrations: $3.14159 = 3.14$ to two decimal places in accordance with (1);

$3.14159 = 3.142$ to three decimal places in accordance with (2);

$3.125 = 3.12$ to two decimal places

and $3.135 = 3.14$ to two decimal places in accordance with (3).

The following examples illustrate the use of the table of square and cube roots as well as the rounding off process.

Example 1. Find the value of each of the following correct to three decimal places: (*a*) $\sqrt{420}$; (*b*) $\sqrt{0.0042}$.

Solution: (*a*) We may write

$$\sqrt{420} = \sqrt{(4.2)(10^2)} = 10\sqrt{4.2}.$$

In Table I opposite 4.2 in the column under the caption $\sqrt{\text{N}}$ we read

$$\sqrt{4.2} = 2.04939.$$

Hence $$10\sqrt{4.2} = 20.4939$$

and $$\sqrt{420} = 20.494,$$ correct to three decimal places. *Ans.*

(*b*) This time we write $\sqrt{0.0042} = \sqrt{\dfrac{42}{10^4}} = \dfrac{\sqrt{42}}{10^2}$.

In the same line of Table I, but under the caption $\sqrt{10\text{N}}$, we find

$$\sqrt{42} = 6.48074.$$

Therefore

$$\frac{\sqrt{42}}{100} = 0.0648074$$

and $$\sqrt{0.0042} = 0.065,$$ correct to three decimal places. *Ans.*

Notice that in problems like the preceding one, we bring a number into the range of our tables by displaying an *even* power of 10 as a multiplier or divisor. It follows from the definition of a square root of a positive number that

$$\sqrt{ab} = \sqrt{a}\sqrt{b},$$

and that

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}.$$

The student may verify these statements by squaring both sides. We have used the last two equations in Example 1.

Example 2. Find the value of each of the following radicals correct to three decimal places: (*a*) $\sqrt[3]{0.56}$; (*b*) $\sqrt[3]{5600}$.

Solution: We use the same idea as in Example 1, employing powers of 10 that are perfect cubes.

(*a*) $$\sqrt[3]{0.56} = \sqrt[3]{\frac{560}{10^3}} = \frac{\sqrt[3]{560}}{10}.$$

In Table I, opposite 5.6 but under the caption $\sqrt[3]{100N}$, we read

$$\sqrt[3]{560} = 8.24257.$$

Thus, $\sqrt[3]{0.56} = 0.824$, correct to three decimal places. *Ans.*

(*b*) $$\sqrt[3]{5600} = \sqrt[3]{(5.6)(10^3)} = 10\sqrt[3]{5.6}.$$

In the same line of Table I, but under the caption $\sqrt[3]{N}$, we find

$$\sqrt[3]{5.6} = 1.77581.$$

Hence

$\sqrt[3]{5600} = 17.758$, correct to three decimal places. *Ans.*

EXERCISES 30

Find the value, correct to three decimal places, of each of the following expressions by using Table I:

1. $\sqrt{1.7}$ **2.** $\sqrt{3.9}$ **3.** $\sqrt{19}$ **4.** $\sqrt{73}$
5. $\sqrt{490}$ **6.** $\sqrt{650}$ **7.** $\sqrt{1300}$ **8.** $\sqrt{5700}$
9. $\sqrt{0.87}$ **10.** $\sqrt{0.38}$ **11.** $\sqrt{0.063}$ **12.** $\sqrt{0.016}$
13. $\sqrt{0.0033}$ **14.** $\sqrt{0.0026}$ **15.** $\sqrt[3]{4.2}$ **16.** $\sqrt[3]{9.7}$
17. $\sqrt[3]{53}$ **18.** $\sqrt[3]{13}$ **19.** $\sqrt[3]{0.46}$ **20.** $\sqrt[3]{0.84}$
21. $\sqrt[3]{170}$ **22.** $\sqrt[3]{350}$ **23.** $\sqrt[3]{2200}$ **24.** $\sqrt[3]{4700}$
25. $\sqrt[3]{63,000}$ **26.** $\sqrt[3]{91,000}$ **27.** $\sqrt[3]{0.018}$ **28.** $\sqrt[3]{0.031}$
29. $\sqrt[3]{450,000}$ **30.** $\sqrt[3]{870,000}$ **31.** $\sqrt[3]{0.0024}$ **32.** $\sqrt[3]{0.0075}$
33. $\sqrt{13} + \sqrt{22} - \sqrt{7}$ **34.** $\sqrt{58} - \sqrt{82} + \sqrt{43}$
35. $4\sqrt{3} - 7\sqrt{2} + 8\sqrt[3]{5}$ **36.** $2\sqrt[3]{19} + 3\sqrt{13} + 5\sqrt{77}$
37. $\sqrt[3]{0.80} - \sqrt{0.41} + \sqrt{0.034}$ **38.** $\sqrt{0.78} - \sqrt[3]{0.0072} + \sqrt{29}$
39. $\sqrt{0.23} + \sqrt[3]{56} - \sqrt{8.5}$ **40.** $\sqrt[3]{0.3} - 2\sqrt{0.002} - \sqrt[3]{0.05}$

42. Exponents

There are many places in mathematics and its applications where it is important to have workable definitions of exponents which are not positive integers. We shall set up these definitions for rational numbers used as exponents.

In our extension of the number system to include the negative numbers, we were guided by the demand that the laws of operation for

the positive numbers should govern operations with the new numbers. Here also, we use a similar principle: *The new exponents shall be so defined that operations with them will be governed by the laws for combining positive integral exponents.*

The definition and the laws for positive integral exponents are listed below for easy reference. (m and n are positive integers.)

Definition: $a^n = a \cdot a \cdot a \cdot \cdots \cdot a$ (n factors).

Laws:

(I) $$a^m \cdot a^n = a^{m+n}.$$

(Ia) $$(a^m)^n = (a^n)^m = a^{mn}.$$

Illustration: $$(a^3)^2 = (a^3)(a^3) = (a^2)(a^2)(a^2) = (a^2)^3 = a^6.$$

(Ib) $$(ab)^m = a^m b^m.$$

Illustration: $$(ab)^2 = (ab)(ab) = a^2b^2.$$

(Ic) $$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}.$$

Illustration: $$\left(\frac{a}{b}\right)^2 = \frac{a}{b} \cdot \frac{a}{b} = \frac{a^2}{b^2}.$$

(II) $$\frac{a^m}{a^n} = \left\{\begin{matrix} a^{m-n}, \text{ if } m > n; \\ 1 \quad, \text{ if } m = n; \\ \dfrac{1}{a^{n-m}}, \text{ if } m < n. \end{matrix}\right\} \quad (a \neq 0)$$

43. Zero as an Exponent

A useful interpretation of zero as an exponent may be obtained by a partial removal of the restriction on the exponents m and n in the first line of Law II. If we agree to use this first line for $m = n$ as well as for $m > n$, we should have, for $a \neq 0$,

$$\frac{a^m}{a^m} = a^{m-m} = a^0.$$

However, by the second line of Law II

$$\frac{a^m}{a^m} = 1.$$

Accordingly, in order to have, for the zero exponent, a meaning which is consistent with the laws for positive integral exponents, we use the following definition:

(D1) **Whenever $a \neq 0$, $a^0 = 1$.**

Illustrations: $(100)^0 = 1;$

$(-20)^0 = 1;$

$(a + b)^0 = 1, \quad \text{if} \quad a + b \neq 0.$

44. Rational Exponents

For greater ease in reading, fractional exponents, which we consider next, are printed here with the *shilling*-type (or slant-bar) fractions; for example, $1/2$ appears in place of $\frac{1}{2}$.

If we use Law Ia in performing the multiplication of $a^{1/2}$ by itself, we find

$$(a^{1/2})^2 = a^{(1/2)(2)} = a.$$

This means that if $a^{1/2}$ is to have a meaning consistent with the laws for positive integral exponents, it should be taken as one of the square roots of a. We shall take it as the *principal* square root, that is,

$$a^{1/2} = \sqrt{a}$$

whenever a is a positive number.

Following the same procedure, we shall have

$$(a^{1/n})^n = a^{(1/n)(n)} = a.$$

This means that $a^{1/n}$ is an nth root of a. Again, we agree on the following definition:

(D2) **Whenever a has a principal nth root, $a^{1/n} = \sqrt[n]{a}$.**

Compare Property (5) in Section 39. The case where a has no real nth root is excluded from the present discussion.

By using Law Ia, we have as the definition of the general positive rational exponent the following:

(D3) $$a^{m/n} = (a^{1/n})^m = (a^m)^{1/n} = (\sqrt[n]{a})^m = \sqrt[n]{a^m},$$

where we assume that the fraction m/n is in its lowest terms. Observe that the numerator of the exponent is the exponent of the power, and the denominator is the index of the radical.

Illustrations: $a^{1/5} = \sqrt[5]{a};$

$$a^{2/5} = \sqrt[5]{a^2} = (\sqrt[5]{a})^2;$$

$$a^{0.3} = a^{3/10} = \sqrt[10]{a^3} = (\sqrt[10]{a})^3.$$

The student should study the following examples carefully to see that he understands in each step what definitions and laws of exponents have been used.

EXAMPLE 1. Evaluate $(10{,}000)^0(16)^{3/4} \div (27)^{4/3}$.

Solution:
$$(10{,}000)^0 = 1;$$
$$(16)^{3/4} = (\sqrt[4]{16})^3 = 2^3 = 8;$$
$$(27)^{4/3} = (\sqrt[3]{27})^4 = 3^4 = 81.$$

Therefore, $(10{,}000)^0(16)^{3/4} \div (27)^{4/3} = \frac{8}{81}$. *Ans.*

EXAMPLE 2. Using principal roots, solve the equation

$$x^{2/3} = 64.$$

Solution: If the exponent $3/2$ is applied to both members of this equation, we have

$$(x^{2/3})^{3/2} = (64)^{3/2},$$

or $x = (\sqrt{64})^3 = 8^3 = 512$. *Ans.*

The check is left to the student.

EXAMPLE 3. Express $(2^{2/3})(4^{4/3})$ as a fractional power of 8.

Solution: Since $2 = 8^{1/3}$, we have

$$2^{2/3} = (8^{1/3})^{2/3} = 8^{2/9};$$
$$4 = 2^2 = (8^{1/3})^2 = 8^{2/3};$$

and $4^{4/3} = (8^{2/3})^{4/3} = 8^{8/9}$.

Thus, $(2^{2/3})(4^{4/3}) = (8^{2/9})(8^{8/9}) = 8^{10/9}$. *Ans.*

EXAMPLE 4. Perform the multiplication indicated by

$$9x^{3/2}y^{1/2} \cdot 3x^{5/2}y^{1/4}.$$

Solution:
$$9x^{3/2}y^{1/2} \cdot 3x^{5/2}y^{1/4} = 27x^{(3/2+5/2)}y^{(1/2+1/4)}$$
$$= 27x^4y^{3/4}. \quad \textit{Ans.}$$

EXAMPLE 5. By factoring $a - b$ as the difference of two squares, remove a factor common to the numerator and denominator of the fraction $\dfrac{a-b}{a^{1/2}+b^{1/2}}$.

Solution: We may think of $a - b$ as $(a^{1/2})^2 - (b^{1/2})^2$ so that

$$\frac{a-b}{a^{1/2}+b^{1/2}} = \frac{(a^{1/2}+b^{1/2})(a^{1/2}-b^{1/2})}{a^{1/2}+b^{1/2}} = a^{1/2} - b^{1/2}. \quad \textit{Ans.}$$

EXERCISES 31

In Problems 1 to 60 all letters represent positive numbers.

In Examples 1 to 8 replace each radical by the equivalent exponential form.

1. $\sqrt{5}$ **2.** $\sqrt[3]{-14}$ **3.** $\sqrt[4]{11}$ **4.** $-\sqrt[3]{ab^2}$

5. $\sqrt[4]{7y^5}$ **6.** $-3\sqrt[5]{8r^4}$ **7.** $\sqrt[3]{-56u^5}$ **8.** $\sqrt[5]{-25x^2}$

In Examples 9 to 17 rewrite each expression in radical form.

9. $5^{3/4}$ **10.** $-(23)^{1/3}$ **11.** $(6x)^{3/2}$

12. $(cu)^{2/3}$ **13.** $(-b)^{2/5}$ **14.** $8y^{4/3}$

15. $3(-7x)^{1/5}$ **16.** $2w^{5/6}v^{1/3}$ **17.** $6a^{1/3}b^{2/3}$

Using principal roots, evaluate each of the expressions in Exercises 18 to 29.

18. $-8^{4/3}$ **19.** $(-8)^{4/3}$ **20.** $625^{3/4}$

21. $(-125)^{4/3}$ **22.** $(-32)^{3/5}$ **23.** $128^{3/7}$

24. $\dfrac{(81)^{3/4}}{8^{2/3}}$ **25.** $\dfrac{4^{3/2}}{-(-64)^{4/3}}$ **26.** $\dfrac{49^{1/2}}{-(-8)^{2/3}}$

27. $(0.0036)^{3/2}$ **28.** $(0.00032)^{4/5}$ **29.** $(-0.000027)^{4/3}$

Perform the multiplications indicated in Problems 30 to 47. Leave all answers in exponential form with exponents simplified as much as possible.

30. $4c^{2/3}d^{1/4} \cdot 2c^{1/6}d^{1/2}$ **31.** $5x^{1/3}y^{2/3} \cdot 3x^{1/4}y^{1/2}$

32. $6a^{7/2}b^{5/2} \cdot 4a^{1/2}b^{3/2}$ **33.** $7u^{3/4}v^{5/6} \cdot 2u^{1/4}v^{1/6}$

34. $(x^2y^5)(2x^2y)^3$ **35.** $(e^2f^3)^4(e^3f)^5$

36. $(2a^{3/7}b)^7(a^{1/2}b^{2/3})^6$ **37.** $(r^{3/8}s^{1/2})^8(r^{4/9}s^{2/3})^9$

38. $(t^{3n}u^n)^f(t^{4f}u^{5f})^n$ **39.** $(w^mz^k)^m(w^kz^m)^k$

40. $(x^{1/2} - y^{1/2})^2$ **41.** $(a^{2/3} + b^{2/3})^2$

42. $(u^{1/3} + 2v^{1/3})^3$ **43.** $(3x^{2/3} - y^{2/3})^3$

44. $(e^{1/3} - f^{1/3})(e^{2/3} + e^{1/3}f^{1/3} + f^{2/3})$

45. $(2a^{5/3} - b^{5/3})(4a^{10/3} + 2a^{5/3}b^{5/3} + b^{10/3})$

46. $(c^{1/2} - d^{1/2})(c^{3/2} + cd^{1/2} + c^{1/2}d + d^{3/2})$

47. $(x^{1/5} + y^{1/5})(x^{4/5} - x^{3/5}y^{1/5} + x^{2/5}y^{2/5} - x^{1/5}y^{3/5} + y^{4/5})$

Simplify the fractions in Examples 48 to 56. Remove all factors common to numerator and denominator, and combine and simplify exponents as much as possible.

48. $\dfrac{12c^{3/2}d^{7/2}}{20c^2d^4}$ **49.** $\dfrac{10a^4b^2}{15a^{5/2}b^{1/2}}$ **50.** $\dfrac{14x^4y^3z^7}{35x^2y^8z^3}$

51. $\dfrac{55a^2b^{1/2}c^{5/6}}{30a^{5/2}bc^{1/3}}$ **52.** $\dfrac{(r^5s^2)^4}{(r^3s^4)^5}$ **53.** $\dfrac{(x^{2/5}y^2)^5}{(x^{1/6}y^{1/3})^{12}}$

54. $\dfrac{(2a^{1/6}b^{2/3})^6}{(-a^{1/4}b^{3/4})^4}$ **55.** $\dfrac{(e^rd^{3s})^4}{(e^{2r}d^s)^3}$ **56.** $\dfrac{(-3a^xb^{2y})^3}{(a^{2x}b^{3y})^4}$

Carry out the divisions indicated in Examples 57 to 60.

57. $\dfrac{9x - 4y}{3x^{1/2} + 2y^{1/2}}$ **58.** $\dfrac{7u - 5v}{\sqrt{7}u^{1/2} - \sqrt{5}v^{1/2}}$

59. $\dfrac{27a^2 - 8b^2}{3a^{2/3} - 2b^{2/3}}$ **60.** $\dfrac{c - d}{c^{1/4} - d^{1/4}}$

In each of the following equations solve for x, using principal roots only:

61. $x^{2/3} = 25$ **62.** $x^{5/3} = -32$ **63.** $x^{3/4} = 125$

64. $8x^{3/2} = 27$ **65.** $x^{7/3} = -128$ **66.** $64x^{6/5} = 1$

45. Negative Exponents

We shall next remove the remaining restriction on the exponents m and n in the first line of Law II by giving a useful meaning to negative exponents. If we agree to use this first line for the division of a^5 by a^8, we should write

$$\frac{a^5}{a^8} = a^{5-8} = a^{-3}.$$

The third line of the same law gives

$$\frac{a^5}{a^8} = \frac{1}{a^3}.$$

A similar procedure may be employed for $\dfrac{1}{a^m}$, where m is any rational number, to give

$$\frac{1}{a^m} = \frac{a^0}{a^m} = a^{0-m} = a^{-m},$$

a result which justifies the following definition:

(D4) ***If*** $a \neq 0$, $a^{-m} = \dfrac{1}{a^m}$.

Illustrations: (*a*) $a^{-\frac{1}{2}} = \dfrac{1}{a^{\frac{1}{2}}} = \dfrac{1}{\sqrt{a}}$;

(*b*) $(32)^{-\frac{3}{5}} = \dfrac{1}{(32)^{\frac{3}{5}}} = \dfrac{1}{(\sqrt[5]{32})^3} = \dfrac{1}{8}$;

(*c*) $10^{-2} = \dfrac{1}{10^2} = \dfrac{1}{100} = 0.01.$

The use of negative exponents makes it possible to write ordinary decimal fractions in a form which displays to advantage the fact that our number system is based on 10. For example, we may write

$$0.637 = 6 \cdot 10^{-1} + 3 \cdot 10^{-2} + 7 \cdot 10^{-3}$$

and $$29.765 = 2 \cdot 10^1 + 9 \cdot 10^0 + 7 \cdot 10^{-1} + 6 \cdot 10^{-2} + 5 \cdot 10^{-3}.$$

The exponents of 10 corresponding to the positions occupied by the digits in the number are the positive integers as we proceed to the left from the units' place and the negative integers as we proceed to the right. The usual manner of writing a number in decimal notation leaves the powers of 10 to be understood. If the two numbers above are multiplied in a manner similar to that used for multinomials, the largest negative exponent of 10 will be -6. Therefore, the multiplication will yield a result with six decimal places. This illustrates a method for obtaining the rules commonly used in combining decimal fractions by means of the fundamental operations.

We shall omit the proof that the new exponents whose meanings have been defined obey all five of the laws which govern positive integral exponents. It is important for the student to convince himself of the reasonableness of the definitions that we have set up, to memorize the definitions carefully, and to practice until he develops a considerable degree of facility in using the new exponents.

Example 1. Simplify $\dfrac{x^{-3}y^2z^{\frac{1}{2}}}{x^2y^{-3}z^{-1}}$ and write the result with positive exponents.

Solution: $$\frac{x^{-3}y^2z^{\frac{1}{2}}}{x^2y^{-3}z^{-1}} = x^{-3-2} \cdot y^{2-(-3)} \cdot z^{(\frac{1}{2})-(-1)}$$

$$= x^{-5}y^5z^{\frac{3}{2}} = \frac{y^5z^{\frac{3}{2}}}{x^5} \cdot \quad \textit{Ans.}$$

Since division by a^m is equivalent to multiplication by $\frac{1}{a^m}$, that is, a^{-m}, it is clear that any *factor* of the entire denominator of a fraction may be written as a factor of the numerator by changing the sign of the exponent of the factor. In a similar way, a factor may be changed from the numerator to the denominator.

Illustrations: (*a*) $\frac{a^{-2}b}{c^{-1}d} = \frac{bc}{a^2d}$, since the factor a^{-2} in the numerator may be written as the factor a^2 in the denominator, and the factor c^{-1} in the denominator may be written as the factor c in the numerator.

(*b*) $$\frac{a(b+c)^2}{xy^2} = a(b+c)^2x^{-1}y^{-2} \quad \text{or} \quad \frac{1}{a^{-1}(b+c)^{-2}xy^2}.$$

EXAMPLE 2. Simplify $\frac{a^{-1}+b^{-1}}{(a+b)^{-1}}$ and write the result using only positive exponents.

Solution: $$\frac{a^{-1}+b^{-1}}{(a+b)^{-1}} = \frac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{a+b}} = \frac{\frac{b+a}{ab}}{\frac{1}{a+b}}$$

$$= \frac{a+b}{ab}(a+b) = \frac{(a+b)^2}{ab}. \quad \textit{Ans.}$$

Notice carefully the distinction between

$$a^{-1}+b^{-1} = \frac{1}{a}+\frac{1}{b} \quad \text{and} \quad (a+b)^{-1} = \frac{1}{a+b}$$

in the last example.

EXAMPLE 3. Express $\left(\frac{x^2y^{-3}}{u^4}\right)^{-\frac{1}{2}}$ in simplest form using positive exponents.

Solution: $$\left(\frac{x^2y^{-3}}{u^4}\right)^{-\frac{1}{2}} = \frac{(x^2)^{-\frac{1}{2}}(y^{-3})^{-\frac{1}{2}}}{(u^4)^{-\frac{1}{2}}} = \frac{x^{-1}y^{\frac{3}{2}}}{u^{-2}}$$

$$= \frac{u^2y^{\frac{3}{2}}}{x}. \quad \textit{Ans.}$$

EXAMPLE 4. Using principal roots, solve the equation

$$3^{x+1} = 9^x \cdot 27^{x+1}.$$

Solution: In a problem of this type, it is desirable to reduce the equation to the form

$$a^u = a^v$$

if possible. Since only principal roots are involved, this equation must imply that $u = v$.

In the given equation, all the bases are powers of 3, so that the required reduction can be made by replacing 9 by 3^2 and 27 by 3^3 as follows:

$$3^{x+1} = (3^2)^x(3^3)^{x+1};$$

or $$3^{x+1} = (3^{2x})(3^{3x+3});$$

and $$3^{x+1} = 3^{5x+3}.$$

Consequently, $$x + 1 = 5x + 3,$$

and $$x = -\tfrac{1}{2}. \quad \textit{Ans.}$$

Check: If $x = -\frac{1}{2}$

Left Member	Right Member
$= 3^{1/2}$	$= (9^{-1/2})(27^{1/2})$
	$= \dfrac{1}{9^{1/2}}(27^{1/2})$
	$= (\frac{27}{9})^{1/2} = 3^{1/2}$

EXERCISES 32

Replace each of Expressions 1 to 8 by an equivalent expression using positive exponents only.

1. $u^{-3}v^{-6}$

2. $x^{-4}y^{-7}$

3. $4a^{-7}b^2$

4. $5k^{-2}r^{-3}$

5. $\dfrac{2}{x^{-5}y^{-4}}$

6. $\dfrac{3}{e^{-6}f^7}$

7. $\dfrac{1}{4(u - v)^{-2}}$

8. $\dfrac{1}{5(3r - 5t)^{-4}}$

Combine exponents to the same base and simplify as much as possible in each of Examples 9 to 18. Give answers with positive exponents only.

9. $(r^6s^{-4}t^4)^0(5r^7s^{-2}t^5)^2$

10. $(g^{-3}h^{-2}k^{-5})(2gh^{-2}k^2)^3$

11. $\dfrac{a^{5/2}b^{-1/6}}{a^{-1/2}b^{4/5}}$ **12.** $\dfrac{x^{-1/4}y^{-5/3}}{x^{7/4}y^{-7/3}}$

13. $\left(\dfrac{c^{-3}d^{6}}{c^{-5}d^{8}}\right)^{-5}\left(\dfrac{c^{-2}d^{-1}}{c^{2}d^{-3}}\right)^{4}$ **14.** $\left(\dfrac{a^{4}b^{-5}}{a^{-2}b^{2}}\right)^{3}\left(\dfrac{a^{-3}b^{-8}}{a^{-5}b^{-6}}\right)^{-4}$

15. $\left(\dfrac{-27u^{-9}v^{6}}{w^{-12}}\right)^{-1/3}$ **16.** $\left(\dfrac{16x^{-8}y^{4}}{z^{20}}\right)^{-1/4}$, $y > 0$, $z > 0$

17. $\left(\dfrac{-5a^{-3}}{21b^{2}}\right)^{-1}\left(\dfrac{49a^{4}}{100b^{-8}}\right)^{-1/2}$ **18.** $\left(\dfrac{81x^{-12}}{y^{16}}\right)^{-1/4}\left(\dfrac{-x^{9}}{27y^{-15}}\right)^{-1/3}$, $x > 0$

In each of Exercises 19 to 22 perform the indicated multiplication. Do not change to positive exponents.

19. $(a^{-4} + b^{-4})(a^{-4} - b^{-4})$

20. $(3x^{-3} + 4y^{-5})(3x^{-3} - 4y^{-5})$

21. $(c^{-1} - y^{-2})(c^{-2} + c^{-1}y^{-2} + y^{-4})$

22. $(a^{-2/3} + 3b^{-2/3})(a^{-4/3} - 3a^{-2/3}b^{-2/3} + 9b^{-4/3})$

Use factoring to simplify each of the fractions in Exercises 23 to 28. Remove factors common to the numerator and the denominator before changing to a form with positive exponents.

23. $\dfrac{x^{-6} + 8y^{-6}}{x^{-2} + 2y^{-2}}$ **24.** $\dfrac{25 - 4w^{-2}}{5w^{-2} - 2w^{-3}}$

25. $\dfrac{36h^{5} - 49h^{-3}}{6 - 7h^{-4}}$ **26.** $\dfrac{8a^{-3} - 27b^{-9}}{2a^{-1}b^{-2} - 3b^{-5}}$

27. $\dfrac{1 - 2y^{-2}}{y^{-1/2} - 2y^{-5/2}}$ **28.** $\dfrac{64a^{-7/3} + a^{-16/3}}{4a^{-1/3} + a^{-4/3}}$

Simplify each of Expressions 29 to 38 as much as you can. Write each result using positive exponents only. Assume that all letters represent positive numbers.

29. $(b^{3+y} \cdot b^{4-y})^{1/7}$ **30.** $(x^{3d-3} \cdot x^{5d+3})^{1/4}$

31. $[(v^{-1} - 1)^{-1} - 1]^{-1}$ **32.** $(x^{-2} + 2)^{-2} - (x^{-2} - 2)^{-2}$

33. $\left(\dfrac{9y^{-4} - x^{-2}}{3y^{-2} - x^{-1}}\right)^{-1}(3x + y^{2})$

34. $[(a - a^{-1})^{2} + 2]^{-1}(a^{-2} + a^{2}) - 1$

35. $\dfrac{1}{1 - 2y^{-1/2}} - \dfrac{1}{1 + 2y^{-1/2}}$ **36.** $\dfrac{c^{-1/2} - 1}{c^{-1/2} + 1} - \dfrac{c^{-1/2} + 1}{c^{-1/2} - 1}$

37. $\sqrt{(e^{y} - e^{-y})^{2} + 4}$ **38.** $\sqrt{(e^{2x} + e^{-2x})^{2} - (e^{2x} - e^{-2x})^{2}}$

Using principal roots only, solve each of Equations 39 to 48 for x.

39. $x^{-3/2} = 27$ **40.** $x^{-2/5} = 4$

41. $x^{-7/3} = -128$ **42.** $4x^{1/3} = -5$

43. $2x^{1/5} = -1$ **44.** $x^{-3/4} = \frac{1}{8}$

45. $2^{3-x} \cdot 4^{2x-1} = 16$

46. $3^{4x-5} \cdot 9^{-2-x} = 243$

47. $5^{2x} \cdot 25 = 125^{x-1}$

48. $2^{2x} \cdot 4^{2x-8} = 8^{x+3}$

The remaining problems in this set are comparatively difficult. Each expression should be put into simplest form using positive exponents. Assume that each letter represents a positive number.

49. $[b^{-3y} \cdot b^{y(y+3)}]^{1/y}$

50. $[x^{4b} \cdot x^{b(b-4)} \cdot x^{-4}]^{1/(b-2)}$

51. $[y^{b-2} \cdot y^{(2+b)(2-b)} \cdot y^{-2-b}]^{1/b}$

52. $(u^{h-m} \cdot u^{m^2} \cdot u^{4m-h})^{1/m}$

53. $[2 - (a + a^{-1})^2]^{-1}(a^{-2} + a^2) + (a^{-2} + a^2)^0$

54. $\dfrac{d^{-1} + e^{-1}}{(d^{-1} - e^{-1})^{-1}} (d^{-2} - e^{-2})^{-1} + 2(d^{-1} + e^{-1})^0$

55. $\{[(c^{-1} - 1)^{-1} + 1]^{-1} - [(c^{-1} + 1)^{-1} - 1]^{-1}\}^{-1}$

56. $\dfrac{x^{-1} - y^{-1}}{x^{-2} + x^{-1}y^{-1} + y^{-2}} + \dfrac{3x^{-1}y^{-1}}{x^{-3} - y^{-3}}$

57. $\left[\dfrac{v^{-1} - 9v}{v^{-1/2} - 3v^{1/2}} - \dfrac{v^{-1} - 1}{v^{-1/2} - v^{1/2}}\right] v^{1/2}$

58. $\dfrac{(32)^{4/5}(2^{n+1})^4}{8^3(16)^{n-1}}$

59. $\dfrac{9^2(27)^{2-n}}{(81)^{-n}(27)^3(3^{n-1})}$

60. $\sqrt[n]{\dfrac{3^{3n+1} + 3^{2n+1}}{3^{2n+1} + 3^{n+1}}}$

61. $\sqrt[n]{\dfrac{64^n + 16^{2n}}{8^n + 32^n}}$

62. $\sqrt[n]{\dfrac{4^n \cdot 6}{4^{2n+1} + 2^{4n+1}}}$

63. $\sqrt[n]{\dfrac{9(3^{4n+1} + 9^{2n})}{3^{2n+2} \cdot 4}}$

Chapter 8

OPERATIONS WITH RADICALS

46. Change of Form

A number of operations with radicals involve changes in form which may be made by the use of the following rules:

(1) $$\sqrt[n]{a^n} = a.$$

(2) $$\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}.$$

(3) $$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}.$$

Unless otherwise indicated, we shall be concerned with real numbers and with principal roots only. The first of the above rules is the equivalent of the definition of the principal nth root of a, as may be seen by changing to the corresponding expression using exponents. Thus,

$$\sqrt[n]{a^n} = a^{n/n} = (a^{1/n})^n = (\sqrt[n]{a})^n = a.$$

The other two rules are obtained from the rules of exponents as follows:

$$\sqrt[n]{ab} = (ab)^{1/n} = a^{1/n} \cdot b^{1/n} = \sqrt[n]{a}\sqrt[n]{b};$$

and $$\sqrt[n]{\frac{a}{b}} = \left(\frac{a}{b}\right)^{1/n} = \frac{a^{1/n}}{b^{1/n}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}.$$

By the use of rules (1) and (2) above, any factor that occurs under the radical sign with an exponent equal to the index of the radical may be written as an outside factor with the exponent 1.

Illustration: $\sqrt[4]{48} = \sqrt[4]{2^4 \cdot 3} = \sqrt[4]{2^4} \cdot \sqrt[4]{3} = 2\sqrt[4]{3}.$

EXAMPLE 1. Write the radical $\sqrt[4]{324a^6b^9}$ with all factors possible removed from under the radical sign.

Solution: $\sqrt[4]{324a^6b^9} = \sqrt[4]{2^2 \cdot 3^4 \cdot a^4 \cdot a^2 \cdot (b^2)^4 \cdot b}$

$$= 3|a|b^2\sqrt[4]{4a^2b}. \quad \textit{Ans.}$$

Since a may be positive or negative, the absolute value bars are necessary to obtain the principal root.

A second useful change of form consists in replacing a factor outside the radical by a corresponding inner factor. Thus,

$$12\sqrt{2} = \sqrt{12^2} \cdot \sqrt{2} = \sqrt{(144)(2)} = \sqrt{288};$$

and $$5\sqrt[3]{2} = \sqrt[3]{5^3} \cdot \sqrt[3]{2} = \sqrt[3]{(125)(2)} = \sqrt[3]{250}.$$

Similarly, any outside factor in a radical expression may be written as an inside factor if an exponent equal to the order of the radical is applied to it.

NOTE: Operations of the type just illustrated are useful in connection with finding roots from tables. For example, in using Table I (see Appendix), the removal of the factor from inside the radical as in the first of the two previous illustrations would make it possible to find $\sqrt{288}$ (approximately) by reading $\sqrt{2}$ and multiplying by 12. With a more extensive table, it would be easier to read $\sqrt{288}$ directly.

EXAMPLE 2. Replace the outside factors of $2x^2y\sqrt[4]{5xy}$ by corresponding inner factors. Assume that x and y are positive.

Solution: $2x^2y\sqrt[4]{5xy} = \sqrt[4]{2^4(x^2y)^4} \cdot \sqrt[4]{5xy}$

$$= \sqrt[4]{(16x^8y^4)(5xy)}$$

$$= \sqrt[4]{80x^9y^5}. \quad \textit{Ans.}$$

EXAMPLE 3. Change the expression $\dfrac{1}{a+b}\sqrt{a^2-b^2}$ to a radical with 1 as its outside coefficient. Assume that $a + b$ is positive.

Solution: $$\frac{1}{a+b}\sqrt{a^2-b^2} = \frac{\sqrt{a^2-b^2}}{\sqrt{(a+b)^2}}$$

$$= \sqrt{\frac{(a-b)(a+b)}{(a+b)^2}}$$

$$= \sqrt{\frac{a-b}{a+b}}. \quad Ans.$$

When a radical has a fractional radicand, it is sometimes convenient to change its form so that either the denominator or the numerator is rational. This procedure is known as **rationalizing** the denominator or the numerator, respectively, and may be accomplished by multiplying numerator and denominator by a properly chosen factor. This factor is the expression of lowest degree and with the least numerical coefficient which will give exponents equal to the index of the part where the rationalization is to be performed.

Example 4. Rationalize the denominator of the radical $\sqrt{\dfrac{3}{32}}$.

Solution: $\sqrt{\dfrac{3}{32}} = \sqrt{\dfrac{3}{2^5}} = \sqrt{\dfrac{3\cdot 2}{2^6}} = \sqrt{\dfrac{6}{(2^3)^2}} = \dfrac{\sqrt{6}}{2^3} = \dfrac{\sqrt{6}}{8}$. *Ans.*

Example 5. Rationalize the numerator of the radical $\sqrt[3]{\dfrac{4x^2y}{5}}$.

Solution:
$$\sqrt[3]{\frac{4x^2y}{5}} = \sqrt[3]{\frac{2^2x^2y}{5}} = \sqrt[3]{\frac{(2^2x^2y)(2xy^2)}{5(2xy^2)}}$$

$$= \sqrt[3]{\frac{2^3x^3y^3}{10xy^2}} = \frac{2xy}{\sqrt[3]{10xy^2}}. \quad Ans.$$

If the entire radicand can be expressed as an exact power whose exponent has a factor in common with the index of the radical, the order of the radical may be lowered as in the following example.

Example 6. Reduce the index of $\sqrt[4]{64x^2y^2}$, assuming that x and y are positive.

Solution:
$$\sqrt[4]{64x^2y^2} = [(8xy)^2]^{1/4} = (8xy)^{2/4}$$

$$= (8xy)^{1/2} = \sqrt{8xy}. \quad Ans.$$

An expression which involves repeated radicals may be simplified by

changing to exponential form. For instance,

$$\sqrt{2\sqrt{2}} = (2 \cdot 2^{1/2})^{1/2} = (2^{3/2})^{1/2} = 2^{3/4}$$
$$= \sqrt[4]{2^3} = \sqrt[4]{8}.$$

A radical is said to be in **standard form** if the following three conditions are satisfied:

(1) *The radicand is positive and has no factors with exponents as great as, or greater than, the index of the radical.*

(2) *The index of the radical is as low as possible.*

(3) *The denominator of the radical is rationalized.*

Thus, the standard form of $\sqrt{8}$ is $2\sqrt{2}$;

that of $\sqrt[3]{\frac{2}{9}}$ is $\frac{\sqrt[3]{6}}{3}$;

and that of $\sqrt[4]{49}$ is $\sqrt{7}$.

Although the standard form is not the simplest for all purposes, answers to the following problems will be given in standard form unless otherwise specified.

EXERCISES 33

Assume that the letters in the following examples represent positive numbers.

Change each of the radicals in Exercises 1 to 18 to the standard form.

1. $\sqrt{98}$ **2.** $\sqrt{108}$ **3.** $\sqrt[3]{81}$
4. $\sqrt[3]{-432}$ **5.** $\sqrt[4]{162}$ **6.** $\sqrt[4]{80}$
7. $\sqrt[5]{-96}$ **8.** $\sqrt{0.72}$ **9.** $\sqrt{0.0050}$
10. $\sqrt{75x^7y^{11}}$ **11.** $\sqrt{360a^4b^{13}}$ **12.** $\sqrt[3]{1024u^7v^9}$
13. $\sqrt[3]{-625a^8b^{10}}$ **14.** $\sqrt[5]{-128x^5y^7}$ **15.** $\sqrt[6]{729a^{10}b^{18}}$
16. $\sqrt[5]{224a^8b^{11}c^{15}}$ **17.** $\sqrt[k]{x^{3k}y^{5k+2}}$ **18.** $\sqrt[n]{c^{5n+3}d^{2n}}$

Absolute value bars should be used in the answers for the next three problems when they are written in standard form.

19. $\sqrt{3x^2 - 18x + 27}$ **20.** $\sqrt{20v - 20v^3 + 5v^5}$
21. $\sqrt{75y^3 - 30y^4 + 3y^5}$

Replace each of the Expressions 22 to 33 by the corresponding equivalent radical with 1 as the outside coefficient.

22. $2\sqrt{5}$ 23. $3\sqrt{6}$ 24. $5\sqrt[3]{3}$

25. $3\sqrt[3]{7}$ 26. $\frac{6y}{7}\sqrt{\frac{35}{72y^2}}$ 27. $\frac{2a}{b^3}\sqrt[3]{\frac{3b^{12}}{4a^2}}$

28. $\frac{3c^2}{5d^3f}\sqrt[3]{\frac{50d^4f^2}{21c^5}}$ 29. $\frac{r}{3st}\sqrt[4]{\frac{162s^7t^3}{r^5}}$

30. $\frac{a^2}{b^2}\sqrt[n]{\frac{b^{2n+2}}{a^{2n-1}}}$ 31. $\frac{x}{y}\sqrt[m]{\frac{y^{m+3}}{x^{m-2}}}$

32. $\frac{5a+b}{3a^2b^2}\sqrt{\frac{18a^4b^4}{25a^2-b^2}}$ 33. $\frac{u-2w}{u+2w}\sqrt{\frac{u^2+4uw+4w^2}{u^2-4w^2}}$

Change each of the remaining radicals in this set to the standard form.

34. $\sqrt{\frac{2}{7}}$ 35. $\sqrt{\frac{8}{3}}$ 36. $\sqrt[3]{\frac{18}{5}}$

37. $\sqrt[3]{\frac{11}{25}}$ 38. $\sqrt[4]{\frac{2}{27}}$ 39. $\sqrt{\frac{3x^5y^3}{8z^7}}$

40. $\sqrt{\frac{54a^3}{75b^7c^5}}$ 41. $\sqrt[3]{\frac{-6s^5}{49r^2}}$ 42. $\sqrt[4]{\frac{32m^7}{125w^9}}$

43. $\sqrt[5]{\frac{-8a^8}{64b^{21}c^{16}}}$ 44. $\sqrt{\frac{2a+7b}{(2a-7b)c^3}}$ 45. $\sqrt[3]{\frac{e^9f^{12}}{e^2+4ef+4f^2}}$

46. $\sqrt[4]{25}$ 47. $\sqrt[4]{36c^2d^{10}}$ 48. $\sqrt[6]{27a^3b^{15}}$

49. $\sqrt[6]{625r^{10}s^4}$ 50. $\sqrt[4m]{a^{2m}b^{10m}}$ 51. $\sqrt[6n]{c^{15n}d^{21n}}$

52. $\sqrt{2\sqrt[3]{2}}$ 53. $\sqrt[3]{25\sqrt{5}}$ 54. $\sqrt[4]{9\sqrt[3]{9}}$

47. Addition and Subtraction of Radicals

We found in the addition and subtraction of algebraic expressions that only like terms can actually be combined; the addition or subtraction of unlike terms must be left in indicated form. The work with radicals involves a corresponding idea.

Two radicals are **identical** only if they have the same index, radicand, and outside coefficient. Radicals are said to be **similar** if they can be changed in form so that their radical factors are identical; in all other cases the radicals are called **dissimilar.** Thus, $\sqrt{50}$ and $\sqrt{72}$ are similar radicals; for

$$\sqrt{50} = 5\sqrt{2},$$

and

$$\sqrt{72} = 6\sqrt{2}.$$

On the other hand, $\sqrt{50}$ and $\sqrt{75}$ are dissimilar radicals; for

$$\sqrt{50} = 5\sqrt{2},$$
$$\sqrt{75} = 5\sqrt{3},$$

and $\sqrt{2}$ and $\sqrt{3}$ cannot be made identical.

In the addition or subtraction of radicals, similar radicals may be combined; the combination of dissimilar radicals can only be indicated. For example, we may perform the operations symbolized by

$$2\sqrt{a} + 5\sqrt{a} - \sqrt{a};$$

the result is $6\sqrt{a}$; but $\sqrt{a} + \sqrt{b}$ and $\sqrt{a} + \sqrt[3]{a}$ must both be left in indicated form.

Example 1. Add $\sqrt{50}$ and $\sqrt[4]{64}$.

Solution: $\sqrt{50} = 5\sqrt{2},$

and $\sqrt[4]{64} = \sqrt[4]{8^2} = \sqrt{8} = 2\sqrt{2}.$

Therefore, $\sqrt{50} + \sqrt[4]{64} = 5\sqrt{2} + 2\sqrt{2} = 7\sqrt{2}.$ *Ans.*

Example 2. Combine the terms in the expression $\sqrt{\frac{1}{2}} - \sqrt{\frac{1}{8}} + \sqrt{\frac{9}{32}}$

Solution:
$$\sqrt{\frac{1}{2}} - \sqrt{\frac{1}{8}} + \sqrt{\frac{9}{32}} = \frac{1}{\sqrt{2}} - \frac{1}{2\sqrt{2}} + \frac{3}{4\sqrt{2}}$$
$$= \frac{1}{\sqrt{2}}\left(1 - \frac{1}{2} + \frac{3}{4}\right)$$
$$= \frac{1}{\sqrt{2}}\left(\frac{5}{4}\right) = \frac{5}{4\sqrt{2}}$$
$$= \frac{5\sqrt{2}}{8}\cdot \quad \textit{Ans.}$$

Example 3. Assuming that a is positive, combine similar radicals in the expression $a\sqrt{27a} + \sqrt{108a^3} - 2\sqrt[3]{16a^4} + a\sqrt[3]{54a}$.

Solution:
$$a\sqrt{27a} + \sqrt{108a^3} - 2\sqrt[3]{16a^4} + a\sqrt[3]{54a}$$
$$= 3a\sqrt{3a} + 6a\sqrt{3a} - 4a\sqrt[3]{2a} + 3a\sqrt[3]{2a}$$
$$= 9a\sqrt{3a} - a\sqrt[3]{2a}$$
$$= a(9\sqrt{3a} - \sqrt[3]{2a}). \quad \textit{Ans.}$$

EXERCISES 34

Change the radicals in each of the following expressions to standard form and combine similar radicals. Assume that all the letters represent positive numbers.

1. $2\sqrt{3} + 4\sqrt{12} - \sqrt{75}$
2. $6\sqrt{2} - 3\sqrt{8} + \sqrt{98}$
3. $3\sqrt{125} - 2\sqrt{45} - 8\sqrt{20}$
4. $2\sqrt{54} - \sqrt{216} - 7\sqrt{24}$
5. $5\sqrt[3]{81} - 3\sqrt[3]{24} + \sqrt[3]{192}$
6. $4\sqrt[3]{16} + 5\sqrt[3]{54} - 2\sqrt[3]{128}$
7. $8df\sqrt{d^2f} - 3\sqrt{d^4f^3} + 5d^2\sqrt{f^3}$
8. $b^2\sqrt{9a^2b^3} - 5b^3\sqrt{25a^2b} + 6a\sqrt{b^7}$
9. $4z^3\sqrt{180z} - 2z^2\sqrt{20z^3} - \sqrt{5z^7}$
10. $3\sqrt{98y^{11}} + y^3\sqrt{162y^5} + 5y^5\sqrt{72y}$
11. $5\sqrt{\dfrac{1}{8}} - 7\sqrt{\dfrac{1}{18}} - 5\sqrt{\dfrac{1}{50}}$
12. $\sqrt{\dfrac{25}{27}} + 6\sqrt{\dfrac{1}{48}} - 2\sqrt{\dfrac{16}{3}}$
13. $\sqrt[3]{\dfrac{81}{16}} - \sqrt[3]{\dfrac{4}{9}} + \sqrt[3]{\dfrac{-1}{144}}$
14. $\sqrt[4]{\dfrac{3}{8}} - \sqrt[4]{\dfrac{32}{27}} - \sqrt[4]{\dfrac{625}{216}}$
15. $\sqrt{\dfrac{r^3}{3s^3}} + rs\sqrt{\dfrac{r}{3s^5}} - \dfrac{s^2}{3}\sqrt{\dfrac{3r^3}{s^7}}$
16. $\sqrt{\dfrac{5a}{b}} + \sqrt{\dfrac{b}{5a}} - \sqrt{\dfrac{5b}{a}}$
17. $\sqrt[3]{5b^5} + b\sqrt[3]{-40b^2} - \sqrt[3]{-135b^5}$
18. $cd^4\sqrt[3]{48c^3d^4} + 2d^3\sqrt[3]{6c^6d^7} + c^2\sqrt[3]{-750d^{16}}$
19. $3\sqrt[3]{2v^7} + \sqrt{12v^5} - v^2\sqrt[3]{-128v} + v\sqrt{48v^3}$
20. $\sqrt[5]{64a^6} + \sqrt{450a^5} - a\sqrt[5]{2a} + a\sqrt{98a^3}$
21. $\sqrt[4]{64} - \sqrt{18} + \sqrt[8]{16}$
22. $\sqrt[3]{375} - \sqrt[6]{576} - 4\sqrt[9]{27}$
23. $(e - f)\sqrt{1 + \dfrac{2f}{e - f}} + \sqrt{36e^2 - 36f^2} - ef\sqrt{\dfrac{9}{f^2} - \dfrac{9}{e^2}}$, $e > f$
24. $2d\sqrt[4]{(c^2d^2 - d^4)^2} + 2c\sqrt{c^4 - c^2d^2} - \sqrt{(2c^2 + 2d^2)(2c^4 - 2d^4)}$, $c > d$

48. Multiplication and Division of Radicals

From rules (2) and (3), page 123, it is clear that two radicals of the *same order* may be multiplied or divided by carrying out the operation under the radical sign.

Illustrations: (*a*) $(\sqrt{2})(\sqrt{5}) = \sqrt{(2)(5)} = \sqrt{10}.$

(*b*) $$\frac{\sqrt[3]{24}}{\sqrt[3]{3}} = \sqrt[3]{\frac{24}{3}} = \sqrt[3]{8} = 2.$$

When the radicals are of different orders, the use of exponents is advised.

EXAMPLE 1. Multiply $\sqrt{2x}$ by $\sqrt[3]{3x^2}$, given that x is positive.

Solution:
$$\begin{aligned}(\sqrt{2x})(\sqrt[3]{3x^2}) &= (2x)^{1/2}(3x^2)^{1/3}\\ &= (2^{1/2})(x^{1/2})(3^{1/3})(x^2)^{1/3}\\ &= (2^{3/6})(x^{3/6})(3^{2/6})(x^{4/6})\\ &= [(2^3)(3^2)(x^3)(x^4)]^{1/6}\\ &= \sqrt[6]{72x^7} = x\sqrt[6]{72x}. \quad \textit{Ans.}\end{aligned}$$

NOTE: In order to write the result of the last example by the use of a single radical, the fractional exponents are first written as equivalent fractions using the LCD of the original exponents.

EXAMPLE 2. Divide $\sqrt[3]{2a}$ by $\sqrt{8a}$, given that a is positive.

Solution:
$$\frac{\sqrt[3]{2a}}{\sqrt{8a}} = \frac{\sqrt[3]{2a}\cdot\sqrt{2a}}{\sqrt{8a}\cdot\sqrt{2a}} = \frac{(2a)^{1/3}(2a)^{1/2}}{4a}$$
$$= \frac{(2a)^{5/6}}{4a} = \frac{\sqrt[6]{32a^5}}{4a}. \quad \textit{Ans.}$$

If the radical expressions $\sqrt{a} + \sqrt{b}$ and $\sqrt{a} - \sqrt{b}$ are multiplied together, the result is the rational expression $a - b$. Each of the first two expressions is called a **rationalizing factor** of the other.

Illustrations: A rationalizing factor of $1 + \sqrt{2}$ is $1 - \sqrt{2}$. A rationalizing factor of $\sqrt{6} - \sqrt{3}$ is $\sqrt{6} + \sqrt{3}$.

EXAMPLE 3. Rationalize the denominator of the fraction $\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$. Find the approximate value of this fraction correct to three decimal places.

Solution: Multiply the numerator and the denominator by $\sqrt{3} + \sqrt{2}$, which is the rationalizing factor of the denominator.

$$\frac{(\sqrt{3}+\sqrt{2})(\sqrt{3}+\sqrt{2})}{(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})} = \frac{3 + 2\sqrt{6} + 2}{3 - 2} = 5 + 2\sqrt{6}.$$

From Table I (Appendix)we read $\sqrt{6} = 2.44949$; therefore,

$5 + 2\sqrt{6} = 9.89898 = 9.899$, correct to three decimal places. *Ans.*

The student should observe that an awkward long division has been avoided by rationalizing the denominator before using the tables.

EXERCISES 35

Assume in this set of exercises that the letters represent positive numbers and that any binomial under a radical sign with an even index represents a positive number.

Perform the multiplications indicated in Examples 1 to 22.

1. $2\sqrt{5} \cdot \sqrt{45}$

2. $\sqrt{63} \cdot 2\sqrt{7}$

3. $\sqrt[3]{3} \cdot \sqrt[3]{9}$

4. $\sqrt[3]{40}\,\sqrt[3]{25}$

5. $\sqrt{\frac{3}{5}} \cdot \sqrt{\frac{2}{9}} \cdot \sqrt{\frac{5}{6}}$

6. $\sqrt{\frac{2}{7}} \cdot \sqrt{\frac{1}{12}} \cdot \sqrt{\frac{14}{3}}$

7. $\sqrt[3]{4} \cdot \sqrt{2}$

8. $\sqrt[4]{3} \cdot \sqrt{2}$

9. $\sqrt{2rs^2t} \cdot \sqrt{7r^3st^5} \cdot \sqrt{14r^6s^3}$

10. $\sqrt{10a^3bc^2} \cdot \sqrt{2ab^2c} \cdot \sqrt{5ab^7c^4}$

11. $4\sqrt{10}(3\sqrt{2} - 2\sqrt{5} + \sqrt{10})$

12. $2\sqrt{21}(3\sqrt{7} - \sqrt{3} - \sqrt{21})$

13. $(\sqrt{5} - 2\sqrt{3})^2$

14. $(\sqrt{10} - 3\sqrt{5})^2$

15. $(\sqrt{16 + 2x} - 2\sqrt{6})^2$

16. $(\sqrt{y + 2} + 2\sqrt{3 + y})^2$

17. $(5\sqrt{2} - 3\sqrt{7})(5\sqrt{2} + 3\sqrt{7})$

18. $(3\sqrt{13} + 4\sqrt{11})(3\sqrt{13} - 4\sqrt{11})$

19. $(4\sqrt{6} - \sqrt{3})(\sqrt{6} + 2\sqrt{3})$

20. $(5\sqrt{2} + 3\sqrt{10})(\sqrt{2} - 2\sqrt{10})$

21. $(\sqrt{7} + \sqrt{5} + \sqrt{3})(\sqrt{7} + \sqrt{5} - \sqrt{3})$

22. $(2\sqrt{10} + \sqrt{2} - \sqrt{7})(2\sqrt{10} - \sqrt{2} + \sqrt{7})$

23. Find the value of $y^2 - 2y - 5$ if $y = 1 - \sqrt{6}$.

24. Find the value of $z^2 - 5z + 3$ if $z = \sqrt{5} - 2$.

25. Find the value of $2x^2 + 7x - 11$ if $x = \dfrac{\sqrt{3} - 7}{2}$.

Give the result of each of the divisions indicated in Examples 26 to 36 in standard radical form.

26. $21 \div 14\sqrt{2}$

27. $15 \div 20\sqrt{5}$

28. $\sqrt[3]{4} \div \sqrt{2}$

29. $\sqrt{27} \div \sqrt[3]{9}$

30. $\sqrt{12} \div \sqrt[4]{2}$

31. $\sqrt{x^7y^2} \div \sqrt{7xy^9}$

32. $\sqrt[3]{-\frac{1}{4}} \div \sqrt{\frac{1}{2}}$

33. $\sqrt[4]{\frac{1}{27}} \div \sqrt{\frac{1}{3}}$

34. $\sqrt[4]{\dfrac{a}{b}} \div \sqrt[3]{\dfrac{b}{a}}$

35. $(\sqrt{14} - \sqrt[3]{21}) \div \sqrt[3]{7}$

36. $(\sqrt{15} + 2\sqrt[3]{9}) \div \sqrt{3}$

Rationalize the denominator in each of Examples 37 to 45.

37. $\dfrac{\sqrt{10}}{\sqrt{5} - 2\sqrt{2}}$ 38. $\dfrac{\sqrt{39}}{\sqrt{13} + \sqrt{3}}$ 39. $\dfrac{2\sqrt{7} - \sqrt{6}}{3\sqrt{6} - 2\sqrt{7}}$

40. $\dfrac{2\sqrt{3} + 4\sqrt{2}}{\sqrt{6} - \sqrt{2}}$ 41. $\dfrac{3\sqrt{6} - \sqrt{30}}{3\sqrt{5} + 2\sqrt{6}}$ 42. $\dfrac{\sqrt{a} + \sqrt{3b}}{\sqrt{3a} - \sqrt{b}}$

43. $\dfrac{c + \sqrt{d^2 + c^2}}{c - \sqrt{d^2 + c^2}}$

44. $\left(\dfrac{\sqrt{x + y} + \sqrt{x}}{\sqrt{x + y} - \sqrt{x}}\right)^{\frac{1}{2}}$

45. $\left(\dfrac{\sqrt{r + 4} - \sqrt{r}}{\sqrt{r + 4} + \sqrt{r}}\right)^{\frac{1}{2}}$

Rationalize the numerator in each of Examples 46 to 50.

46. $\dfrac{2\sqrt{3} - \sqrt{2}}{5\sqrt{6}}$ 47. $\dfrac{\sqrt{7} + \sqrt{5}}{\sqrt{7} - \sqrt{5}}$ 48. $\dfrac{2\sqrt{2} + \sqrt{7}}{15 + 4\sqrt{14}}$

49. $\dfrac{\sqrt{x + y} - \sqrt{x}}{y}$ 50. $\dfrac{\sqrt{a + b} + \sqrt{a - b}}{\sqrt{a + b} - \sqrt{a - b}}$

Rationalize the denominator of each of the following fractions, and then use Table I to find the value of the fraction correct to three decimal places:

51. $\dfrac{\sqrt{2} + 2\sqrt{5}}{\sqrt{5} - \sqrt{2}}$ 52. $\dfrac{\sqrt{3} - 3\sqrt{5}}{4\sqrt{3} + 3\sqrt{5}}$ 53. $\dfrac{2\sqrt{3} + \sqrt{2}}{\sqrt{6} + 2\sqrt{2}}$

54. $\dfrac{3}{\sqrt{3.5} - \sqrt{1.5}}$ 55. $\dfrac{\sqrt{3} - 4}{(1 + \sqrt{3})(\sqrt{3} - 2)}$ 56. $\dfrac{\sqrt{2} + \sqrt{3}}{(\sqrt{2} + 1)(3\sqrt{2} - 2)}$

49. Equations Involving Radicals

An equation which contains the unknown involved under one or more radicals can sometimes be solved by the process of rationalization explained in the next examples. The fundamental idea is that, if equal numbers are raised to the same positive, integral power, the results are equal. Of course, the assumption is made that there exists one or more values of the unknown for which the members of the given equation are equal; this assumption may be false in some cases.

EXAMPLE 1. Solve the equation $\sqrt{x - 11} = \sqrt{x} - 1$.

Solution: Square both members

$$x - 11 = (\sqrt{x} - 1)^2,$$

or
$$x - 11 = x - 2\sqrt{x} + 1.$$
$$2\sqrt{x} = 12.$$
$$\sqrt{x} = 6.$$
Square both members
$$x = 36. \quad \textit{Ans.}$$

The student should check this result.

EXAMPLE 2. Solve the equation $\sqrt{2x} + 1 = \sqrt{2x - 3}$.

Solution: Square both members
$$(\sqrt{2x} + 1)^2 = 2x - 3,$$
or
$$2x + 2\sqrt{2x} + 1 = 2x - 3.$$
Hence,
$$2\sqrt{2x} = -4;$$
$$\sqrt{2x} = -2.$$
Square both members
$$2x = 4,$$
or
$$x = 2. \quad \textit{Ans.}$$

Check: If $x = 2$

Left Member	Right Member
$= \sqrt{4} + 1$	$= \sqrt{1} = 1$
$= 3$	Does not check

Therefore, the given equation has no root.

The student should keep in mind that only principal roots are to be used because of our definition of the symbol $\sqrt{a}$. Consequently, the investigation of an equation like that in Example 2 may be stopped at the point in the analysis where $\sqrt{2x} = -2$, because a negative number is not the principal square root of any number.

The restriction to principal roots is an advantage rather than a disadvantage (as the student may think). For example, we may distinguish between the two equations
$$\sqrt{2x} + 1 = \sqrt{2x - 3}$$
and
$$\sqrt{2x} - 1 = \sqrt{2x - 3}.$$

The solution that results from squaring both members gives the same trial value, namely, $x = 2$, for both; but this value checks only in the second equation.

EXERCISES 36

Solve and check each of the following equations:

1. $\sqrt{x} = \sqrt{x + 16} - 2$
2. $\sqrt{z + 4} + 1 = \sqrt{z + 11}$
3. $\sqrt{3y + 1} - 1 = \sqrt{3y - 8}$
4. $\sqrt{1 - u} + 2 = \sqrt{13 - u}$
5. $\sqrt{v + 7} = 5 + \sqrt{v - 2}$
6. $\sqrt{w + 10} - 4 = \sqrt{w - 3}$
7. $\sqrt{7x - 6} = \sqrt{7x + 22} - 2$
8. $\sqrt{18 - 5y} = 4 + \sqrt{-6 - 5y}$
9. $\sqrt{5 - 33w} + 3 = \sqrt{56 - 33w}$
10. $\sqrt{4x + 11} + 5 = \sqrt{4x + 46}$
11. $\sqrt{39 - t} + \sqrt{11 - t} = 6$
12. $\sqrt{5 - 2z} + \sqrt{7 - 2z} = 4$
13. $\sqrt{x + 2c} - \sqrt{c} = \sqrt{x - 3c}$
14. $\sqrt{10a - 3y} = \sqrt{-5a - 3y} + 3\sqrt{a}$
15. $\sqrt{4v + 1} = \sqrt{v + 4} + \sqrt{v - 3}$
16. $\sqrt{12x + 1} = \sqrt{3x + 3} + \sqrt{3x - 2}$
17. $\sqrt{5y + 16c} + \sqrt{5y - 11c} = \sqrt{20y + c}$
18. $\sqrt{3x - 5e} + \sqrt{3x + 19e} = 2\sqrt{3x + 6e}$

50. Problems Involving Radicals and Fractional Exponents

Practical problems frequently involve the use of radicals, as is indicated in the next two examples.

Example 1. Two guy wires extend to the ground from the top of a vertical pole of height h ft. If one wire is fastened to a stake in the ground a ft from the foot of the pole and the second wire is fastened to a similar stake $2a$ ft from the foot of the pole, find a formula for the difference in the lengths of the two wires.

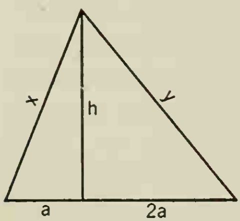


Fig. 32

Solution: Let x = the number of feet in the length of the shorter wire,

and y = the number of feet in the length of the longer wire.

Figure 32 shows the pole and the two wires. We use the fact that the square of the hypotenuse of a right triangle equals the sum of the squares of the two legs.

From the triangle on the left, we find

$$x^2 = a^2 + h^2,$$

or

$$x = \sqrt{a^2 + h^2}.$$

(The negative square root has no significance in this problem.) Similarly, from the triangle on the right, we get

$$y^2 = 4a^2 + h^2,$$

or

$$y = \sqrt{4a^2 + h^2}.$$

Hence, the desired formula is

$$y - x = (\sqrt{4a^2 + h^2} - \sqrt{a^2 + h^2}) \text{ ft.} \quad \textit{Ans.}$$

EXAMPLE 2. Express the area A of one face of a cube in terms of the volume V.

Solution: Let the edge of the cube be e units long. Then,

$$V = e^3 \qquad \text{and} \qquad A = e^2.$$

From the first equation, we find $e = V^{1/3}$. Hence, we may replace e in the second equation by $V^{1/3}$ to obtain

$$A = V^{2/3}. \quad \textit{Ans.}$$

EXERCISES 37

1. Express the circumference C of a circle in terms of its area A.
2. Obtain a formula for the area A of an equilateral triangle in terms of a side s.
3. Express the area A of a regular hexagon in terms of a side s.
4. What is the volume V of a sphere in terms of its surface area S?
5. Find a formula for the area A of an equilateral triangle in terms of its altitude h.
6. What is the volume V of a cube in terms of its body diagonal d?
7. A regular pyramid has a square base. If the side of the base is $2a$ and the lateral edge is $3a$, find the altitude and volume of the pyramid.
8. The area of an equilateral triangle is 144 sq in. Determine the altitude h and the side s.
9. The perimeter of a regular hexagon is 24 in. What is its area?
10. The difference of two positive numbers is 27, and the difference of their square roots is 3. Find the numbers.
11. Find two positive numbers which differ by 33 and which have 11 as the sum of their square roots.

12. The hypotenuse of a right triangle is $m^2 + n^2$ units long and one of the legs is $m^2 - n^2$ units long. What is the length of the other leg?

13. Express the altitude h and the volume V of a regular tetrahedron (a triangular pyramid with equal edges) in terms of an edge e.

14. If the legs of an isosceles right triangle are increased by 2 ft and 5 ft, respectively, there is formed a right triangle whose hypotenuse is 5 ft longer than that of the original triangle. Find the dimensions of the first triangle.

15. If one leg of a right triangle is increased by 2 ft and the other leg is decreased by 7 ft, there is formed an isosceles right triangle whose hypotenuse is 5 ft shorter than that of the original triangle. Find the dimensions of the isosceles triangle.

51. Complex Numbers

In our discussion of the real number system, we made the statement that if n is even, a negative real number has no real nth root. In other words, if we restrict ourselves to real numbers only, the symbols $\sqrt{-2}$, $\sqrt[4]{-16}$, $\sqrt[6]{-12}$, and so forth, are meaningless. However, because of the important part that such symbols can be made to play, not only in theoretical mathematics, but also in practical applications, it is desirable to give them a meaning. The only way in which this can be accomplished is by another extension of the number system of algebra.

The unit of real numbers, the number 1, has two square roots, $+1$ and -1. In order to have a complete analogy with this situation, *we define the two square roots of* $\mathbf{-1}$ *to be* $\mathbf{+i}$ *and* $\mathbf{-i}$, *and agree always to replace the symbol* $\sqrt{\mathbf{-1}}$ *by the symbol* $\mathbf{i}$, *and* $\mathbf{i^2}$ *by* $\mathbf{-1}$. The number i is called the **imaginary unit**.

Remarks: (1) In electric-circuit analysis, engineers customarily use the symbol j in place of i.

(2) The traditional choice of the adjectives *real* and *imaginary* to describe numbers in algebra is most unfortunate. These adjectives are used here as technical terms to distinguish two kinds of numbers, neither of which is more "imaginary" than the other.

The symbol $\sqrt{-a}$, where a is a *positive* number, may now be written $\sqrt{(-1)(a)} = (\sqrt{-1})(\sqrt{a}) = i\sqrt{a}$. Since $\sqrt{a}$ is a real number, the square roots of any negative number may be expressed as products of the imaginary unit by a real number; this procedure should always be followed in order to avoid confusion.

Illustrations: (a) The square roots of -4 are $+2i$ and $-2i$.

(b) The square roots of -10 are $i\sqrt{10}$ and $-i\sqrt{10}$.

Any number of the form $a + bi$, where a and b are *real* numbers,

is a **complex number.** The a is the *real part* and the bi is the *imaginary part* of the complex number; b is the coefficient of the imaginary part.

Illustrations: $2 + 3i$, $-3 + 4i$, $0 - 2i$, and $2 + 0i$ are all complex numbers.

If $a = 0$ and $b \neq 0$ in the number $a + bi$, the number is a **pure imaginary number.** If $b = 0$, the number is a real number. The complex numbers, then, include both real and pure imaginary numbers as special cases. We shall use the term **imaginary number** to mean that $b \neq 0$.

The ordinary rules of addition, subtraction, multiplication, and division apply to complex numbers. (A discussion of this statement is postponed to more advanced work in algebra.)

Example 1. Add the complex numbers $2 + \sqrt{-4}$, $3 + \sqrt{-25}$, and $-6 - \sqrt{-9}$.

Solution:

$$\begin{aligned} 2 + \sqrt{-4} &= 2 + 2i \\ 3 + \sqrt{-25} &= 3 + 5i \\ -6 - \sqrt{-9} &= -6 - 3i \\ \hline \text{The sum} &= -1 + 4i. \quad \textit{Ans.} \end{aligned}$$

Example 2. Multiply $2 + 3i$ by $5 - 4i$.

Solution:

$$\begin{aligned} (2 + 3i)(5 - 4i) &= 10 + 7i - 12i^2 \\ &= 10 + 7i - (12)(-1) \\ &= 22 + 7i. \quad \textit{Ans.} \end{aligned}$$

It should be noted that the multiplication is carried out as usual for binomials, then -1 is put in place of i^2 wherever it occurs.

Example 3. Show that $i^6 = -1$.

Solution: $i^6 = (i^2)^3 = (-1)^3 = -1$. *Ans.*

The two numbers $a + bi$ and $a - bi$ are **conjugate complex numbers.** Thus,

$$2 + 3i, \qquad 2 - 3i;$$

$$-4 - 7i, \qquad -4 + 7i;$$

and

$$4i, \qquad -4i$$

are three pairs of conjugate complex numbers.

The fact that the product of two conjugate complex numbers is real, that is,

$$(a + bi)(a - bi) = a^2 + b^2,$$

can always be used, as in the next example, to write a fraction, having an imaginary denominator, in the standard form $a + bi$.

EXAMPLE 4. Write the quotient of $2 + 3i$ divided by $3 + 2i$ in the form $a + bi$.

Solution: $\dfrac{2 + 3i}{3 + 2i} = \dfrac{(2 + 3i)(3 - 2i)}{(3 + 2i)(3 - 2i)}$

$$= \frac{6 + 5i - 6i^2}{9 - 4i^2} = \frac{12 + 5i}{13} = \frac{12}{13} + \frac{5}{13}i. \quad Ans.$$

EXERCISES 38

Perform the indicated operations and reduce each of Expressions 1 to 24 to the form $a + bi$.

1. $(9 + \sqrt{-16}) + (2 + \sqrt{-25}) + (6 - \sqrt{-64})$

2. $(13 - \sqrt{-36}) - (10 - \sqrt{-49}) + (8 + \sqrt{-4})$

3. $(-8 + \sqrt{-2}) - (-5 + \sqrt{-32}) - (15 - 3\sqrt{-18})$

4. $(10 + \sqrt{-50}) + (11 - \sqrt{-20}) - (6 + \sqrt{-80})$

5. $(\sqrt{3} + i2\sqrt{5}) - (4\sqrt{3} - i4\sqrt{5}) + (8\sqrt{3} + i3\sqrt{5})$

6. $(\sqrt{10} - i\sqrt{63}) - (-3\sqrt{10} - i6\sqrt{7}) - (5\sqrt{10} + i\sqrt{112})$

7. $(5 - 2i)(1 - 4i)$ **8.** $(2 + 3i)(3 - i)$

9. $(7 - 3i)(-2 + 5i)$ **10.** $(-6 - 4i)(5 + 4i)$

11. $(\sqrt{5} + i2\sqrt{3})(2\sqrt{5} - i3\sqrt{3})$ **12.** $(3\sqrt{7} - i\sqrt{6})(\sqrt{7} - i2\sqrt{6})$

13. $i^6 - i^7 - i^{10}$ **14.** $(i^2)^3 - (i^5)^4 + i^{17}$

15. $(2 - i)^2 + (3 + 3i)^2$ **16.** $(6 - 2i)^2 - (4 - i)^2$

17. $(2 + i)^3 - (2 - i)^3$ **18.** $(-1 - i)^4$

19. $(3 - i) \div (1 - 2i)$ **20.** $(7 - 26i) \div (3 - 4i)$

21. $\dfrac{13 + 26i}{2 + 3i}$ **22.** $\dfrac{10 + 15i}{6 - 8i}$

23. $\dfrac{7 + i14\sqrt{3}}{2 - i\sqrt{3}}$ **24.** $\dfrac{27 - i9\sqrt{2}}{4 + i\sqrt{2}}$

25. Show that $-3 + 5i$ and $-3 - 5i$ are roots of $y^2 + 6y + 34 = 0$.

26. Show that $6 + 2i$ and $6 - 2i$ are roots of $z^2 - 12z + 40 = 0$.

27. Show that $1 - 2i$ is a root of $\dfrac{5}{x} - \dfrac{5}{x - 2} = 2$.

28. Show that $2 + 3i$ is a root of $\dfrac{13}{y-4} - \dfrac{13}{y} = -4$.

29. Show that $4 - i$ is a root of $1 + \dfrac{2}{v-3} = \dfrac{5}{v-2}$.

EXERCISES 39

Change Expressions 1 to 4 to simplest form with positive exponents only. Assume that the letters represent positive numbers.

1. $\left(\dfrac{2\sqrt[3]{a^2b^{-2}c^{-4}}}{\sqrt[3]{64a^{-2}b^{-3}c^9}}\right)^{-3}$ **2.** $\left(\dfrac{\sqrt[4]{9r^{-2}s^3t^{-5}}}{3rs^2\sqrt{t}}\right)^{-8}$

3. $[4 + (2a - a^{-1})^2]^{-1} \div [4a^2 + a^{-2}]^{-1}$

4. $5(a - b)^{-2} - 3(ab)^{-2}(b^{-1} - a^{-1})^{-2}$

Simplify each of the following four radicals as much as you can:

5. $\sqrt{32^5 + 2^{25}}$ **6.** $\sqrt[3]{-81^2 - 3^8}$

7. $\sqrt{16^2 + 2^8 + 4^4 + 256}$ **8.** $\sqrt[4]{3^4 + 9^2 + 81}$

Find the positive square root of each of the following four expressions. [HINT: $a + b + 2\sqrt{ab} = (\sqrt{a} + \sqrt{b})^2$.]

9. $10 - 2\sqrt{21}$ **10.** $13 + 2\sqrt{42}$

11. $8 + \sqrt{48}$ **12.** $11 - \sqrt{112}$

Rationalize the denominator of each of the following five fractions:

13. $\dfrac{2 + \sqrt{3} - \sqrt{7}}{2 - \sqrt{3} - \sqrt{7}}$ **14.** $\dfrac{\sqrt{5} + \sqrt{3} - 2\sqrt{2}}{\sqrt{5} + \sqrt{3} + 2\sqrt{2}}$ **15.** $\dfrac{12}{\sqrt{3 - \sqrt{3}}}$

16. $\dfrac{1}{1 - \sqrt[3]{h}}$ [HINT: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.]

17. $\dfrac{1}{2\sqrt[3]{a} + \sqrt[3]{b}}$ (See the preceding problem.)

Solve each of the following equations for y and check:

18. $(3^{4y-3})^{-2} = 27^{-y-8}$ **19.** $(a^3)^{2y-6} = (a^{-2})^{4-y}$, $a > 0$

20. $h^2(w + y)^{2/3} = m^2$ **21.** $n^2z^{3/2}y^{-2/3} = r^6$

22. $\dfrac{5}{\sqrt{4y-1}} - \sqrt{4y - 1} = 2\sqrt{y}$

23. $\sqrt{25 + 12y} - \dfrac{13}{\sqrt{25 + 12y}} = 2\sqrt{3y}$

Chapter 9

QUADRATIC EQUATIONS IN ONE UNKNOWN

52. The Standard Quadratic Form

A rational integral equation that contains the unknown to the second (and no higher) degree is called a **quadratic** equation. If x is written for the unknown number, such an equation may always be put into the *standard quadratic form*

$$ax^2 + bx + c = 0,$$

where the coefficients a, b, and c do not depend on x. If $b = 0$, the equation is a **pure** quadratic equation.

Illustrations: (a) The equation $(2x + 3)^2 = (x + 8)(x + 5)$ may be put in standard quadratic form by performing the indicated multiplications and collecting the terms. Thus we obtain

$$4x^2 + 12x + 9 = x^2 + 13x + 40;$$

which becomes $\qquad 3x^2 - x - 31 = 0.$

In this equation, by comparison with the standard quadratic form, $a = 3$, $b = -1$, and $c = -31$.

(b) $8x^2 - 9 = 0$ is a pure quadratic equation. Here, $a = 8$, $b = 0$, and $c = -9$.

Pure quadratic equations may always be solved by isolating x^2 on one side of the equation; the roots are then found by extracting the square roots of both members.

Example 1. Solve the equation $4x^2 - 25 = 0$.

Solution:

Add 25 to both members $\quad 4x^2 = 25.$

Divide by 4 $\quad x^2 = \frac{25}{4}.$

Consequently, $\quad x = \pm\sqrt{\frac{25}{4}} = \pm\frac{5}{2}.$ *Ans.*

NOTE: The sign $\pm$ is read "plus or minus." It is easy to verify the fact that nothing is gained by writing $\pm x = \pm\frac{5}{2}$.

EXERCISES 40

Write each of Equations 1 to 6 in the form $ax^2 + bx + c = 0$.

1. $6y^2 - y + 15 = 4 - y(y + 3)$
2. $x(x - 4) + 11 = (3x - 8)x$
3. $(v - 6)^2 + (3v + 2)^2 = (2v - 1)^2$
4. $(2u - 4)^2 - (2 - u)^2 = 5u^2$
5. $z(3z + 6k) + 10k = 2z(z - 5k) - z^2 + 6$
6. $4x(x + m) + x(5m - 6x) = 8m - 3$

For each of Equations 7 to 10 find the equation in standard quadratic form which may be obtained by freeing the given equation of fractions.

7. $\dfrac{u + 5}{2u} - \dfrac{u}{2u - 3} = \dfrac{3}{4}$
8. $\dfrac{3y + 7}{y - 1} - \dfrac{y + 2}{2} = \dfrac{5}{6}$
9. $\dfrac{3}{x + 2} - \dfrac{5}{x + 1} = \dfrac{2}{x - 3}$
10. $\dfrac{1}{z - 4} + \dfrac{3}{3 - 2z} = \dfrac{6}{z + 7}$

Each of the following equations is reducible to the pure quadratic form in one of the unknowns x, y, or z. Solve each equation by first making this reduction.

11. $3x(x - 8) = 4x(2x - 6) - 15$
12. $5z(2z - 3) + 35 = 3z(z - 5)$
13. $6y(y + 2a) = (3y + 2a)^2$
14. $(x + 2c)^2 = 4x(c + x)$
15. $\dfrac{7}{9 - x^2} - \dfrac{5}{6} = \dfrac{27}{54 - 6x^2}$
16. $\dfrac{8 - 9y^2}{3y^2 - 4} + \dfrac{5}{2} = \dfrac{2y^2 - 4}{12 - 9y^2}$
17. $\sqrt{2x + 10} = \sqrt{x - 12} + \sqrt{x + 12}$
18. $\sqrt{6y - 16} + \sqrt{3y - 6} = \sqrt{3y + 6}$

53. Solution by Factoring

The number system of algebra possesses the important property: *If the product of two numbers is zero, at least one of the numbers must be zero.*

In symbols, we have

if $$AB = 0,$$

then $$A = 0,$$

or $$B = 0,$$

or $$A = 0 \quad \text{and} \quad B = 0.$$

This property is the key to the solution of equations by the method of factoring; for, if we can factor the left member of a quadratic equation *in standard form,* we can find its roots at once by solving the two linear equations obtained by putting each factor equal to zero. It is important to note that the right member of the equation *must be zero* before we may apply this principle. Any quadratic equation that is not in standard form should be put into that form before applying the method of factoring.

EXAMPLE 1. Solve the equation $5x^2 + 2x - 3 = 0$.

Solution: By factoring the left side, we have

$$(5x - 3)(x + 1) = 0.$$

At least one of these factors must be zero. If

$$5x - 3 = 0,$$

then, $$x = \tfrac{3}{5}. \quad \textit{Ans.}$$

If $$x + 1 = 0,$$

then, $$x = -1. \quad \textit{Ans.}$$

EXAMPLE 2. Solve the equation $x^2 + 5x = 0$.

Solution: We factor the left member to get

$$x(x + 5) = 0.$$

Then, $$x = 0. \quad \textit{Ans.}$$

Or $$x + 5 = 0,$$

and $$x = -5. \quad \textit{Ans.}$$

The student may verify both of these roots. Observe that if the members of the equation had been divided by x, the root $x = 0$ would have been lost.

EXERCISES 41

Solve each of the following equations for the appropriate letter x, y, or z by the method of factoring:

1. $3x^2 - 8x = 0$

2. $5z^2 + 7z = 0$

3. $z^2 - z - 12 = 0$

4. $y^2 - 9y + 18 = 0$

5. $y(3y + 11) = 20$

6. $x(6x + 7) = 3$

7. $4x^2 - 20x + 25 = 0$

8. $9z^2 + 48z + 64 = 0$

9. $y^2 - 4b^2 - y - 2b = 0$

10. $x^2 - 9c^2 - x + 3c = 0$

11. $z^2 - 4z + 4 = 9c^2d^2$

12. $9y^2 - 6y - 16a^2 + 1 = 0$

13. $x^2 + 2ax = 8a^2$

14. $z^2 - 2bz = 15b^2$

15. $7y^2 - 11h^2y = 6h^4$

16. $12x^2 - 10m^2x = 12m^4$

17. $z^3 + 64z = 0$

18. $y^3 + y^2 - 12y = 0$

19. $x^3 + 8b^3 - 2bx(x + 2b) = 0$

20. $27z^3 - c^3 + 3cz(3z - c) = 0$

21. $9y^2 - c^2 + 4cd - 4d^2 = 0$

22. $25y^2 - a^2 - 9b^2 - 6ab = 0$

23. $z^2 + 4d^2 - z + 2d = 4dz$

24. $b^2x^2 - 4bx + 3 = 4x^2 - 4x$

54. Solution by Completing the Square

We shall illustrate by examples the method of completing the square. This method will be used in the next section to obtain a formula for finding the roots of any quadratic equation.

EXAMPLE 1. Solve the equation $x^2 - 10x - 24 = 0$.

Solution: Add 24 to both sides so that the terms in x are isolated.

$$x^2 - 10x = 24.$$

Since

$$(x \pm a)^2 = x^2 \pm 2ax + a^2,$$

it appears that in order to complete the trinomial square when the first two terms $x^2 \pm 2ax$ are given, we must add a^2. Since

$$a^2 = [\tfrac{1}{2}(2a)]^2,$$

the term to be added is the square of half the coefficient of x.

In the equation above, half the coefficient of x is -5; hence, $(-5)^2 = 25$ is added to both sides of the equation to give

$$x^2 - 10x + 25 = 24 + 25,$$

or

$$(x - 5)^2 = 49.$$

Extract the square roots of both members to get

$$x - 5 = \pm 7.$$

Therefore, $x = 5 \pm 7 = 12$ or -2. *Ans.*

EXAMPLE 2. Solve the equation $3x^2 + 4x - 1 = 0$.

Solution:

Add 1 to both members	$3x^2 + 4x = 1.$
Divide both sides by 3	$x^2 + \frac{4}{3}x = \frac{1}{3}.$
Add $(\frac{2}{3})^2 = \frac{4}{9}$ to both sides	$x^2 + \frac{4}{3}x + \frac{4}{9} = \frac{1}{3} + \frac{4}{9},$
or	$(x + \frac{2}{3})^2 = \frac{7}{9}.$
Extract square roots	$x + \frac{2}{3} = \pm\frac{1}{3}\sqrt{7}$
and	$x = \frac{1}{3}(-2 \pm \sqrt{7}).$ *Ans.*

Check: If $x = \dfrac{-2 + \sqrt{7}}{3}$

Left Member	Right Member
$= 3\left(\dfrac{-2 + \sqrt{7}}{3}\right)^2 + 4\left(\dfrac{-2 + \sqrt{7}}{3}\right) - 1$	$= 0$
$= 3\left(\dfrac{11 - 4\sqrt{7}}{9}\right) - \dfrac{8}{3} + \dfrac{4\sqrt{7}}{3} - 1$	
$= \dfrac{11}{3} - \dfrac{4\sqrt{7}}{3} - \dfrac{8}{3} + \dfrac{4\sqrt{7}}{3} - 1 = 0$	

The check for $x = \dfrac{-2 - \sqrt{7}}{3}$ may be obtained from the above check by replacing $\sqrt{7}$ by $-\sqrt{7}$.

EXERCISES 42

Solve each of the following equations by the method of completing the square:

1. $x^2 - 2x - 24 = 0$
2. $v^2 + 5v - 6 = 0$
3. $y^2 - 6y + 7 = 0$
4. $z^2 + 10z + 18 = 0$
5. $3u^2 = 6 - 7u$
6. $2x^2 = 56 - 9x$
7. $7z^2 - 11z - 6 = 0$
8. $6t^2 + 11t - 10 = 0$
9. $16y^2 - 24y = -5$
10. $9x^2 - 6x = 8$
11. $x^2 + 10x + 19 = 0$
12. $4z^2 - 4z - 9 = 0$

13. $9v^2 - 30v + 21 = 0$ **14.** $25w^2 - 10w - 8 = 0$

15. $4r = 3r^2 + 5$ **16.** $3z = 2z^2 + 7$

55. The Quadratic Formula

The method of completing the square may be carried out in perfectly general form as follows. Let the equation be

$$ax^2 + bx + c = 0. \qquad (a \neq 0)$$

Subtract c from both sides $\quad ax^2 + bx = -c.$

Divide both sides by a $\quad x^2 + \frac{b}{a}x = -\frac{c}{a}.$

Add $\left(\frac{b}{2a}\right)^2$ to both sides $\quad x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2},$

or $\quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$

Extract square roots $\quad x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}.$

Therefore,

$$\boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}.$$

This is the well-known *quadratic formula.* It should be memorized and thoroughly understood by every student of algebra. It is easily shown by direct substitution that the values of x given by this formula actually do satisfy the equation.

EXAMPLE 1. Solve the equation $2x^2 - 5x + 3 = 0$.

Solution: In this equation, $a = 2$, $b = -5$, and $c = 3$. By the quadratic formula

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(2)(3)}}{2(2)} = \frac{5 \pm \sqrt{1}}{4}$$

$$= 1 \text{ or } \tfrac{3}{2}. \quad \textit{Ans.}$$

EXAMPLE 2. Solve the equation $x^2 + 2x + 2 = 0$.

Solution: By the formula

$$x = \frac{-2 \pm \sqrt{4 - 8}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2}$$

$$= -1 \pm i. \quad \textit{Ans.}$$

Check: If $x = -1 + i$

Left Member	Right Member
$= (-1 + i)^2 + 2(-1 + i) + 2$	$= 0$
$= 1 - 2i - 1 - 2 + 2i + 2 = 0$	

The root $x = -1 - i$ may be checked in a similar manner.

EXAMPLE 3. The equation $Lm^2 + Rm + \frac{1}{C} = 0$ occurs in electric-circuit analysis. Solve for m in terms of L, R, and C.

Solution: $a = L$, $b = R$, and $c = \frac{1}{C}\cdot$

Therefore,
$$m = \frac{-R \pm \sqrt{R^2 - \frac{4L}{C}}}{2L}$$
$$= -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}\cdot \quad \textit{Ans.}$$

NOTE: The constants R, L, and C are usually such that $\frac{R^2}{4L^2}$ is less than $\frac{1}{LC}\cdot$ In this case, the roots are preferably written

$$m = -\frac{R}{2L} \pm j\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\cdot \quad (j = \sqrt{-1})$$

EXERCISES 43

Each of the equations in this set is to be solved by the quadratic formula.

1. $v^2 + 4v - 60 = 0$

2. $x^2 - 3x - 28 = 0$

3. $5z^2 - 13z + 6 = 0$

4. $12y^2 + 5y - 2 = 0$

5. $u^2 + 2u - 2 = 0$

6. $x^2 + 6x + 10 = 0$

7. $28z^2 + 13z - 6 = 0$

8. $27y^2 = 6y + 8$

9. $x^2 = 6x - 4$

10. $4z(z - 5) = -29$

11. $53 = v(12 - 9v)$

12. $1 = 8x(x + 1)$

13. $25x^2 + 19 = -40x$

14. $9z^2 + 6z - 1 = 0$

15. $7u^2 - 3u + 2 = 0$

16. $4v^2 + 4 = 7v$

17. $5 = 16x - 4x^2$

18. $5y^2 - 1 = 5y$

19. $z^2 + 2\sqrt{3}z - 1 = 0$

20. $\sqrt{2}x^2 + 5x + 2\sqrt{2} = 0$

21. $\sqrt{5}x^2 + 4x + \sqrt{5} = 0$

22. $3r^2 + \sqrt{10}r + 1 = 0$

The following equations are to be solved for the letter which occurs in each case after the equation:

23. $ax^2 + 2bx + c = 0; \quad x$

24. $s^2z^2 - 2sz - m = 0; \quad s$

25. $2b^2y^2 - by + n = 0; \quad b$

26. $r^2x^2 + 5rx + 2h = 0; \quad r$

27. $v^2 + 5v = 6k - k^2; \quad k$

28. $3c^2 + cx - 2x^2 = 2c; \quad c$

29. $s = \frac{1}{2}gt^2 + v_0t; \quad t$

30. $Ma^2 + 2Ra + K = 0; \quad a$

31. Solve $x^2 + 3xy - y^2 - 4 = 0$ for x in terms of y; for y in terms of x.

32. Solve $x^2 - xy - 2y^2 + 5 = 0$ for x in terms of y; for y in terms of x.

56. Factoring by Solving Quadratic Equations

If we permit the use of complex numbers, the *quadratic function* can always be factored. We use the method of completing the square.

$$
\begin{aligned}
ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) \\
&= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a}\right) \\
&= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right] \\
&= a\left[\left(x + \frac{b}{2a}\right)^2 - \left(\frac{\sqrt{b^2 - 4ac}}{2a}\right)^2\right] \\
&= a\left(x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right) \\
&= a\left[x - \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\right]\left[x - \left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)\right].
\end{aligned}
$$

The fractions inside the inner parentheses are the roots of the equation $ax^2 + bx + c = 0$. If these roots are denoted by r_1 and r_2, respectively, we may write

$$ax^2 + bx + c = a(x - r_1)(x - r_2).$$

Example 1. Factor $3x^2 + 5x + 6$.

Solution: We write the corresponding quadratic equation $3x^2 + 5x + 6 = 0$ and find the roots to be $\dfrac{-5 + i\sqrt{47}}{6}$ and $\dfrac{-5 - i\sqrt{47}}{6}$.

Hence,

$$3x^2 + 5x + 6 = 3\left(x - \frac{-5 + i\sqrt{47}}{6}\right)\left(x - \frac{-5 - i\sqrt{47}}{6}\right). \quad \textit{Ans.}$$

EXERCISES 44

Factor each of the following expressions by the method of this section:

1. $2x^2 - x - 28$
2. $12h^2 - h - 6$
3. $10z^2 - 23z + 12$
4. $8v^2 - 2v - 15$
5. $8 + 10w - 7w^2$
6. $15 - 8x - 12x^2$
7. $5z^2 - 3z - 1$
8. $3u^2 - 7u - 2$
9. $7y^2 - y + 2$
10. $6w^2 - 2w + 3$
11. $16r^2 + 8r - 5$
12. $10v^2 - v + 2$
13. $6x^2 - xy - 12y^2$
14. $15u^2 - 2uv - 77v^2$
15. $10b^2c^2 - 17bcd + 3d^2$
16. $18x^2 - 9xyz - 35y^2z^2$
17. $(3b + 9)x^2 + (b^2 + 2b - 9)x - 2b + 2$
18. $(c^2 - 4)y^2 - (3c^2 + 5c + 2)y - 3c$

57. The Graph of the Quadratic Function; Maximum and Minimum Values

The graph of the *quadratic function* when a, b, c are real numbers can be found by writing

$$y = ax^2 + bx + c$$

and proceeding as in Section 30. The graph of a quadratic function is called a **parabola.**

Figure 33 is the graph of the particular quadratic function $y = x^2 - 4x + 2$. We can investigate the shape of this parabola by finding its intersections with lines parallel to the X axis. Each of these lines has an equation of the form $y = k$, where k is the distance from the X axis to the line. We put $y = k$ in the equation of the parabola to obtain

$$x^2 - 4x + (2 - k) = 0.$$

Solving by formula, we find

$$x = \frac{4 \pm \sqrt{16 - 4(2 - k)}}{2}$$

$$= 2 \pm \sqrt{k + 2}.$$

If the quantity $k + 2$, which occurs under the radical, is negative, the corresponding values of x will be imaginary and there will be no points of intersection of the line $y = k$ with the parabola. Now, $k + 2$ is negative if k is negative and numerically greater than 2. Hence, any horizontal line that lies below the line $y = -2$ has no intersection with the parabola.

If $k + 2 = 0$, that is, if $k = -2$, the value of the radical is zero and we get the one value $x = 2$. This means that the line $y = -2$ just touches (is tangent to) the parabola at the point $(2, -2)$.

Finally, if $k + 2$ is positive, there are two values of x; the corresponding points are equidistant from the line $x = 2$ (see Figure 33). Therefore, the curve is symmetrical with respect to the vertical line $x = 2$. This line, which is called the **axis** of the parabola, passes through the lowest point, the **vertex,** of the curve.

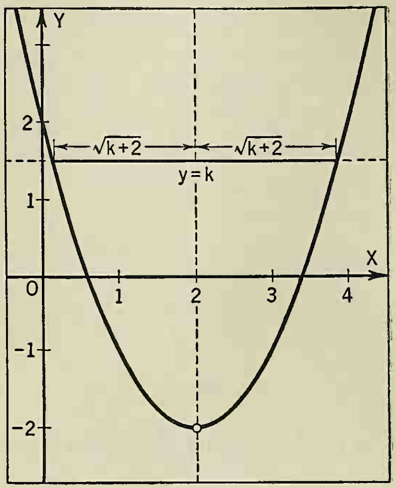


Fig. 33

This same procedure can be carried out in the general case for $y = ax^2 + bx + c$. We find that the parabola opens upward if a is positive, and downward if a is negative. The vertex (which is either the lowest or the highest point on the curve) always has the x coordinate $-\dfrac{b}{2a}$, the first term of the quadratic formula, and the equation of the axis is $x = -\dfrac{b}{2a}$.

EXAMPLE 1. Find the coordinates of the vertex and make a graph of the parabola $2y = 2 - 3x - x^2$.

Solution: We write the equation in standard form

$$y = -\tfrac{1}{2}x^2 - \tfrac{3}{2}x + 1.$$

The axis of the parabola is the line

$$x = -\frac{b}{2a} = -\frac{-\frac{3}{2}}{-1} = -\frac{3}{2}.$$

To find the vertex, we put $x = -\frac{3}{2}$ in the equation of the curve

$$y = -\tfrac{1}{2}(-\tfrac{3}{2})^2 - \tfrac{3}{2}(-\tfrac{3}{2}) + 1$$
$$= -\tfrac{9}{8} + \tfrac{9}{4} + 1 = \tfrac{17}{8}.$$

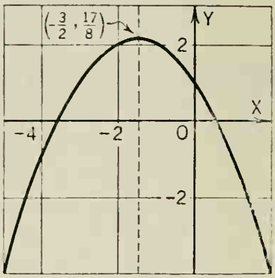


Fig. 34

Therefore, the vertex is at the point $(-\frac{3}{2}, \frac{17}{8})$. The graph is shown in Figure 34. The parabola opens downward because a is negative. *Ans.*

It has been noted before that the points where the X axis intersects the graph of an equation of the type $y = f(x)$ furnish us with roots of the equation $f(x) = 0$. The graphical interpretation of the nature of the roots of the equation

$$ax^2 + bx + c = 0$$

is made clear by the following analysis. We substitute

$$x = -\frac{b}{2a}$$

into the equation $y = ax^2 + bx + c$

to obtain

$$y = a\left(-\frac{b}{2a}\right)^2 + b\left(-\frac{b}{2a}\right) + c$$
$$= \frac{b^2}{4a} - \frac{b^2}{2a} + c$$
$$= -\frac{b^2}{4a} + c$$
$$= -\frac{b^2 - 4ac}{4a}.$$

If we assume a is positive so that the parabola opens upward, we find that the y coordinate of the vertex is

(1) negative, if $b^2 - 4ac$ is positive;
(2) zero, if $b^2 - 4ac = 0$;
(3) positive, if $b^2 - 4ac$ is negative.

The three cases are shown in Figure 35. In the first case, the roots of the equation $ax^2 + bx + c = 0$ are *real and distinct*, and the parabola is cut in *two points* by the X axis. In the second case, the roots are said to be *real and equal;* the parabola just *touches* the X axis. In the third

case, the roots are *imaginary*, and this corresponds to the fact that the parabola does not intersect the X axis.

It is helpful to visualize what would happen to the roots if the parabola starts from a position where its vertex is below the X axis and moves upward until its vertex is on the axis and then passes above it. The points of intersection with the axis would move toward each other, become coincident, and then disappear. This is one reason for saying that the equation $ax^2 + bx + c = 0$ has two equal roots, rather than only one root, when $b^2 - 4ac = 0$.

There are certain simple problems in which it is useful to know the maximum or minimum value of a quadratic function. The following example furnishes such an instance.

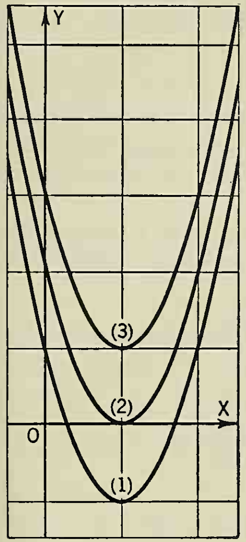


Fig. 35

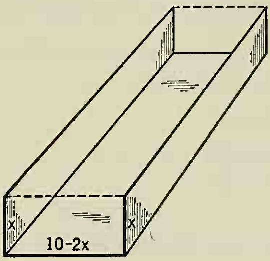


Fig. 36

Example 2. A long rectangular piece of tin 10 in. wide is to be bent into a rectangular channel, as shown in Figure 36. How deep should the channel be if its cross-sectional area is to be as large as possible?

Solution: Let the depth of the channel be x in. Then $(10 - 2x)$ in. is the width. If the cross-sectional area is y sq in.,

$$y = x(10 - 2x) = -2x^2 + 10x.$$

This is the equation of a parabola which opens downward and has the line $x = \frac{5}{2}$ as its axis. The vertex is the highest point, so the largest value of y occurs on the axis of the parabola. The depth of the channel should therefore be 2.5 in. *Ans.*

(This answer will give a cross-sectional area of 12.5 sq in.) The student may check by making a graph of the function.

EXERCISES 45

Find the coordinates of the vertex of the parabola represented by each of Equations 1 to 8, and sketch the curve.

1. $y = x^2 - 12x + 27$

2. $y = 4x^2 - 24x + 35$

3. $4y = x^2 + 4x + 12$

4. $2y = 5x^2 + 10x + 13$

5. $y = 12 + 4x - x^2$

6. $y = 10x - x^2 - 17$

7. $x = y^2 - 8y + 18$

8. $3x = 2y^2 + 16y + 17$

9. Find two numbers whose sum is 18 and whose product is a maximum.

10. The lower base of a trapezoid is 4 in. longer than the upper base, and the sum of the two bases and the altitude is 36 in. Find the dimensions of the trapezoid in order that the area be a maximum.

11. There are 500 yd of fencing available for three sides of a rectangular field adjoining a straight river bank. If no fence is needed along the river, find the largest area that can be enclosed.

Solve each of the following problems on pages 84–85 by the method of this section:

12. Problem 27

13. Problem 28

14. Problem 29

15. Problem 30

16. Problem 31

17. Problem 32

58. The Character of the Roots

We found in Section 55 that the two roots r_1 and r_2 of the quadratic equation $ax^2 + bx + c = 0$ are given by the expressions

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

and

$$r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

We shall restrict the discussion here to quadratic equations with real coefficients, that is, a, b, and c are to be taken as *real* numbers. In this case, the character of the roots is determined by the quantity under the radical sign in the formulas for r_1 and r_2. This quantity $\boldsymbol{b^2 - 4ac}$ is called the **discriminant** of the quadratic equation. It is clear that

(1) If $b^2 - 4ac$ is positive, the roots are real and unequal;

(2) If $b^2 - 4ac$ is zero, the roots are real and equal;

(3) And if $b^2 - 4ac$ is negative, the roots are conjugate imaginary.

Illustrations: (*a*) Consider the nature of the roots of the equation

$5x^2 - 7x + 2 = 0$. For this equation,

$$b^2 - 4ac = 49 - 40 = 9.$$

Hence, the roots are real and unequal. The student may check this result by finding the roots, which are $\frac{2}{5}$ and 1.

(*b*) For the equation $4x^2 - 4x + 1 = 0$,

$$b^2 - 4ac = 16 - 16 = 0.$$

Hence, the roots are real and equal.

(*c*) For the equation $2x^2 - x + 2 = 0$,

$$b^2 - 4ac = 1 - 16 = -15.$$

Hence, the roots are conjugate imaginary.

It may be noted further that, if a, b, and c are rational numbers, the two roots of the quadratic equation will be rational if, and only if, the discriminant is a perfect square. In the first of the preceding illustrations, $b^2 - 4ac$ is a perfect square, and the equation has rational roots.

Example 1. Show that $x^2 + 3x + 4$ is positive for all real values of x.

Solution: Let $y = x^2 + 3x + 4$. The graph of this equation is a parabola with a vertical axis. Also the discriminant of the quadratic function is

$$b^2 - 4ac = 9 - 16 = -7.$$

Hence, y cannot be zero for any real value of x. In other words, the parabola does not cross the X axis. Furthermore, for some particular choice of x, such as $x = 0$, y has the positive value 4. Therefore, y is positive for all real values of x.

Example 2. Show that the function $2x^2 + xy + y^2$ is positive or zero for all real values of x and y.

Solution: We see first that the function is zero if $x = y = 0$. If $y \neq 0$, we may write

$$2x^2 + xy + y^2 = y^2\left[2\left(\frac{x}{y}\right)^2 + \left(\frac{x}{y}\right) + 1\right].$$

Now, let $$\frac{x}{y} = u.$$

Then, $$2\left(\frac{x}{y}\right)^2 + \left(\frac{x}{y}\right) + 1 = 2u^2 + u + 1$$

is a quadratic function whose value is positive if $u = 0$ and whose discriminant is

$$b^2 - 4ac = 1 - 8 = -7.$$

Hence, the quadratic function of u is positive for all real values of u. Furthermore, if $y = 0$ and $x \neq 0$, then the function becomes $2x^2$, which is clearly positive. It therefore follows that the given function is positive or zero for all real values of x and y.

Following the procedure of the two preceding examples, we may show that any quadratic function $ax^2 + bxy + cy^2$, where a, b, and c are real, is positive or zero for all real values of x and y if its discriminant is negative and a is positive.

EXERCISES 46

For each of Equations 1 to 10 determine by an examination of the discriminant whether the roots are real and unequal, real and equal, or imaginary. If the roots are real, state also whether they are rational or irrational.

1. $2y^2 + 10y + 3 = 0$

2. $3x^2 - 4x + 7 = 0$

3. $4z^2 - 36z + 81 = 0$

4. $5v^2 + 8v - 21 = 0$

5. $2x^2 - 10x + 17 = 0$

6. $2w^2 + 2\sqrt{6}\,w + 3 = 0$

7. $v^2 - 0.6v - 0.16 = 0$

8. $x(x + 0.5) + 0.7 = 0$

9. $\sqrt{5}\,s^2 - 4s + \sqrt{5} = 0$

10. $7r^2 + 1 = r(3r + 2)$

Show by an examination of the discriminant that each of the following functions is positive or zero for all values of x and y:

11. $4x^2 + 6xy + 5y^2$

12. $2x^2 - xy + 3y^2$

13. $x^2 + 6xy + 9y^2 + 2$

14. $x^2 + 4xy + 5y^2 + 1$

59. Properties of the Roots

If the roots of the quadratic equation $ax^2 + bx + c = 0$ are added, we have

$$r_1 + r_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{-2b}{2a} = -\frac{b}{a}.$$

Hence, *the sum of the roots of a quadratic equation is the negative of the ratio of the coefficient of* x *to the coefficient of* x²; *that is,*

$$r_1 + r_2 = -\frac{b}{a}. \tag{1}$$

If we multiply the roots together, we find

$$r_1 \cdot r_2 = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$$

$$= \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}.$$

Therefore, *the product of the roots of a quadratic equation is the ratio of the constant term to the coefficient of* x^2*; that is,*

$$\boldsymbol{r_1 r_2 = \frac{c}{a}}. \qquad (2)$$

EXAMPLE 1. Without solving the equation, find the sum and the product of the roots of $5x^2 = 7x - 3$.

Solution: The equation, in standard form, is

$$5x^2 - 7x + 3 = 0,$$

so that $a = 5$, $b = -7$, and $c = 3$.

Using the properties above, we find the *sum* of the roots is

$$\frac{-b}{a} = \frac{-(-7)}{5} = \frac{7}{5}; \quad \textit{Ans.}$$

and the *product* of the roots is

$$\frac{c}{a} = \frac{3}{5}. \quad \textit{Ans.}$$

The student may check these results by solving the equation and finding the sum and the product of the roots.

EXAMPLE 2. Write the quadratic equation whose roots are $2 + \sqrt{3}$ and $2 - \sqrt{3}$.

Solution: If $r_1 = 2 + \sqrt{3}$ and $r_2 = 2 - \sqrt{3}$,

then, $$r_1 + r_2 = 4.$$

Hence, $$-\frac{b}{a} = 4.$$

Also $$r_1 r_2 = 4 - 3 = 1,$$

so that $$\frac{c}{a} = 1.$$

The members of $ax^2 + bx + c = 0$ may be divided by a, to obtain

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$

Using the values just found for $\frac{b}{a}$ and $\frac{c}{a}$, we have

$$x^2 - 4x + 1 = 0. \quad \textit{Ans.}$$

This result may be verified by actually solving the equation and checking with the given roots.

Let us consider a second approach to this problem. In Section 56, we factored the quadratic function $ax^2 + bx + c$ to obtain

$$a(x - r_1)(x - r_2).$$

Therefore, for given roots r_1 and r_2, the corresponding quadratic equation must be

$$a(x - r_1)(x - r_2) = 0.$$

Or, since $a \neq 0$,

$$(x - r_1)(x - r_2) = 0.$$

If the roots are $2 + \sqrt{3}$ and $2 - \sqrt{3}$, we have

$$[x - (2 + \sqrt{3})][x - (2 - \sqrt{3})] = 0$$

and

$$(x - 2)^2 - 3 = 0,$$

or

$$x^2 - 4x + 1 = 0$$

as before.

Example 3. Determine the value of k so that one root of

$$x^2 + 2x + k = 0$$

shall be double the other root.

Solution: Let r_1 and r_2 be the roots of the equation. Then, for the sum of the roots, we must have

$$r_1 + r_2 = -\frac{b}{a} = -2,$$

and, since one root is to be double the other,

$$r_1 = 2r_2.$$

Solving the system of linear equations for r_1 and r_2, we find $r_1 = -\frac{4}{3}$ and $r_2 = -\frac{2}{3}$. Since $r_1 r_2 = \dfrac{c}{a} = k$, we have

$$k = (-\tfrac{2}{3})(-\tfrac{4}{3}) = \tfrac{8}{9}. \quad Ans.$$

The check is left for the student.

EXERCISES 47

Use Properties (1) and (2) described in the preceding section to find the sum and product of the roots of each of the following equations:

1. $y^2 - 16y + 40 = 0$ **2.** $x^2 + 11x - 18 = 0$

3. $v(5v - 7) - 8 = 0$ **4.** $p(7p - 12) = 14$

5. $10 = 3x(x + 6)$ **6.** $5 = v(6 - 3v)$

7. $(a + b)x^2 + (a^3 + b^3)x + 6ab = 0$

8. $(c^2 - d^2)y^2 + (c + d)y + c - d = 0$

9. $x^2 - 6h = h(10 + 3x)$ **10.** $my(y + 6m) = 17$

For each of the following examples, form a quadratic equation that has the two given numbers for its roots. In each case, use both of the methods employed in Example 2 of the preceding section.

11. $-5, 7$ **12.** $3, 8$ **13.** $-4, -9$

14. $4 + \sqrt{5}, 4 - \sqrt{5}$ **15.** $-2 + 3\sqrt{2}, -2 - 3\sqrt{2}$

16. $-3 + 2i, -3 - 2i$ **17.** $5 + 8i, 5 - 8i$

18. $6 + i\sqrt{2}, 6 - i\sqrt{2}$ **19.** $-3 + i2\sqrt{5}, -3 - i2\sqrt{5}$

60. Roots of Special Quadratics

We shall consider in this section the roots of the equation

$$ax^2 + bx + c = 0, a \neq 0,$$

if one or both of the coefficients c and b are zero.

(1) *If* b $= 0$ *and* c $\neq 0$, *the quadratic equation has two roots which are equal numerically but are opposite in sign.*

Since the equation is $ax^2 + c = 0$, we have

$$x = \pm\sqrt{\frac{-c}{a}}.$$

(2) *If* c $= 0$ *and* b $\neq 0$, *the quadratic equation has one zero root.*

Under the given conditions, the equation is

$$ax^2 + bx = 0,$$

or
$$x(ax + b) = 0.$$

Hence,
$$x = 0 \quad \text{or} \quad x = -\frac{b}{a}\cdot$$

(3) *If* c = 0 *and* b = 0, *both roots of the quadratic equation are zero.* In this case, the equation is $ax^2 = 0$, which clearly has two zero roots.

EXAMPLE 1. For what values of m does the equation

$$x^2 + m^2x = (2m + 3)x + 4$$

have roots that are equal numerically but are opposite in sign?

Solution: The equation may be written in standard form

$$x^2 + (m^2 - 2m - 3)x - 4 = 0.$$

Then, by using (1), we must have

$$b = m^2 - 2m - 3 = 0,$$

that is,
$$m = 3 \quad \text{or} \quad m = -1. \quad \textit{Ans.}$$

For either of these values of m, the roots of the equation are $x = 2$ and $x = -2$.

EXAMPLE 2. For what values of m and n will the equation

$$x^2 + (m + n - 6)x + (2m - 5n - 5) = 0$$

have two zero roots?

Solution: Using condition (3) of this section, we must have $b = 0$ and $c = 0$; hence,

$$m + n - 6 = 0,$$

and
$$2m - 5n - 5 = 0.$$

This system of linear equations is satisfied by the pair of values $m = 5$, $n = 1$. *Ans.*

EXERCISES 48

For each of the following equations, find for what value or values of m one root is zero:

1. $m^2y^2 - 2y + 5m - 8 = 0$ **2.** $7u^2 + (4 - m)u + 6m - 10 = 0$

3. $3x^2 - 14 + m^2 = 4mx - 5m$ **4.** $w^2 + 30 = 6w + m(3m + 13)$

For each of the following equations, find for what value or values of k the roots are equal numerically but are opposite in sign:

5. $z^2 + 7kz - 10 = 25z$
6. $5x^2 + 4kx = k^2 + 12 - 18x$
7. $6x^2 + (2k^2 - 7)x = 19 + 5kx$
8. $4u^2 + (6k^2 + 29k)u = 22k + 20$

For each of the following equations, find for what value or values of k and m both roots are zero:

9. $11x^2 + (5m - 3k - 6)x + 3m - k = -4$
10. $5mv^2 + (2m + 4k)v + 4m + 16k = 3v + 7$
11. $7my^2 - (6k + 17)y - (1 - 2k) = -m(y + 3)$
12. $(3k + m)w^2 + (2m - 3k)w + 4m = 11w + 5k + 13$

Find the value of a, b, c, or k in each of the following equations for the given condition:

13. $6y^2 + by - 4 = 0$; one root is $\frac{1}{3}$.
14. $7x^2 + bx - 2 = 0$; one root is $\frac{1}{7}$.
15. $3z^2 + 7z + c = 0$; one root is -4.
16. $y^2 - 4y + c = 0$; one root is $2 - \sqrt{7}$.
17. $az^2 = 17z - 6$; one root is $\frac{2}{5}$.
18. $ax^2 + 15 = 26x$; one root is $\frac{3}{4}$.
19. $v^2 - 9v + c = 0$; one root exceeds the other by 5.
20. $8x^2 - 46x + c = 0$; one root exceeds the other by $\frac{3}{4}$.
21. $(k - 4)y^2 + 2ky - 11 = 0$; the sum of the roots is -3.
22. $(3k - 1)x^2 = 10 - 5kx$; the sum of the roots is 4.
23. $w^2 - 17w + c = 0$; the difference of the roots is 11.
24. $z^2 + 11z + 3k - 6 = 0$; the difference of the roots is 7.
25. $9x^2 - 15x + c = 0$; one root is 4 times the other.
26. $5y^2 + 18y + c = 0$; the quotient of the roots is 5.
27. $3v^2 - 14v + 2k + 4 = 0$; the quotient of the roots is 6.
28. $(2k + 6)w^2 - (8k + 2)w + 5k = 0$; the product of the roots is $\frac{15}{8}$.
29. $(k - 1)x^2 - (4k + 3)x = 4 - 6k$; the product of the roots is $\frac{20}{3}$.
30. $2ky^2 + (3k + 4)y + 6 = 0$; one root is the reciprocal of the other.
31. Show that the roots of $ax^2 - bx - c = 0$ are the reciprocals of the roots of $cx^2 + bx - a = 0$.

61. Equations That Lead to Quadratic Equations

The student should be sure he understands the methods used for simplifying equations involving fractions and equations involving radicals before proceeding to the examples which follow.

EXAMPLE 1. Solve the equation

$$\frac{x-2}{x^2-x-6} - \frac{x}{x^2-4} = \frac{3}{2(x+2)}.$$

Solution: Multiply both members by the LCD, which is

$$2(x-2)(x+2)(x-3),$$

to obtain

$$2(x-2)^2 - 2x(x-3) = 3(x-2)(x-3),$$

or $$2x^2 - 8x + 8 - 2x^2 + 6x = 3x^2 - 15x + 18.$$

Hence, $$3x^2 - 13x + 10 = 0,$$

or $$(3x-10)(x-1) = 0,$$

and $$x = 1 \text{ or } \tfrac{10}{3}. \quad \textit{Ans.}$$

As the student can verify, both of these values are roots of the original equation. When in the process of simplifying a given equation the two members are multiplied by an expression involving the unknown, it is possible that the resulting equation will possess roots that are not roots of the original equation. Consequently, it is urgent in such a case that all suspected roots be checked in the given equation.

EXAMPLE 2. Solve the equation $\sqrt{6x+1} + \sqrt{x} = \sqrt{10x+9}$.

Solution: Square both sides

$$6x + 1 + 2\sqrt{6x^2+x} + x = 10x + 9.$$

Subtract $7x + 1$ from both sides

$$2\sqrt{6x^2+x} = 3x + 8.$$

Square both sides $$24x^2 + 4x = 9x^2 + 48x + 64.$$

Therefore, $$15x^2 - 44x - 64 = 0,$$

or $$(x-4)(15x+16) = 0,$$

and $$x = 4 \text{ or } -\tfrac{16}{15}.$$

Check: If $x = 4$

Left Member	Right Member
$= \sqrt{25} + \sqrt{4}$	$= \sqrt{49} = 7$
$= 5 + 2 = 7$	

If $x = -\frac{16}{15}$

Left Member	Right Member
$= \sqrt{-\frac{96}{15} + 1} + \sqrt{-\frac{16}{15}}$	$= \sqrt{-\frac{160}{15} + 9}$
$= \sqrt{-\frac{81}{15}} + \sqrt{-\frac{16}{15}}$	$= \sqrt{-\frac{25}{15}}$
$= \frac{9i}{\sqrt{15}} + \frac{4i}{\sqrt{15}}$	$= \frac{5i}{\sqrt{15}}$
$= \frac{13i}{\sqrt{15}}$	Does not check.

Therefore, $x = 4$ is the only root of the given equation. The original equation and the equation obtained after squaring each member may not be equivalent, as seen above. It is demonstrable that squaring each member of an equation does not cause a loss of roots; unfortunately, however, the new equation thus obtained may have roots that are not solutions of the original equation. Consequently, it is always necessary to check all suspected roots obtained by such a process; those that do not satisfy the given equation are frequently characterized as extraneous.

EXERCISES 49

Factor the left member of each of Equations 1 to 6, and then solve by any convenient method.

1. $y^3 + 64 = 0$ **2.** $27x^3 - 8 = 0$ **3.** $125z^3 + 1 = 0$

4. $x^4 - 16 = 0$ **5.** $64y^6 - 1 = 0$ **6.** $216v^3 - 125 = 0$

7. Solve for $\frac{y}{x}$: $21x^2 + 23xy - 20y^2 = 0$

8. Solve for $\frac{x}{y}$: $20x^2 + 7xy - 6y^2 = 0$

Solve each of the following equations for x, y, or z by any convenient method:

9. $\dfrac{y-1}{y-2} - \dfrac{y-3}{y-4} = -\dfrac{2}{15}$ **10.** $\dfrac{x+2}{x+1} + \dfrac{x-1}{x-3} = \dfrac{19}{6}$

11. $\dfrac{2z+11}{z+4} + \dfrac{z-2}{z-4} = \dfrac{7}{2} + \dfrac{12}{z^2-16}$

12. $\dfrac{4y+3}{2y+1} - \dfrac{18}{5} = \dfrac{4y^2+4}{4y^2-1} + \dfrac{2y-3}{2y-1}$

13. $\dfrac{x-4}{2x^2+5x-3} - \dfrac{4x-1}{4x^2+13x+3} + \dfrac{2x+7}{8x^2-2x-1} = 0$

14. $\dfrac{5z-8}{2z^2-13z+15} - \dfrac{4z+4}{3z^2-13z-10} - \dfrac{6z+4}{6z^2-5z-6} = 0$

15. $\sqrt{2x-2} + \sqrt{x-5} = 2\sqrt{x}$

16. $\sqrt{3-y} + \sqrt{1-4y} = \sqrt{4-10y}$

17. $\sqrt{x+1} + \sqrt{x+8} = \sqrt{5x+9}$

18. $\sqrt{z-1} + \sqrt{2z+6} = \sqrt{7z+1}$

19. $\sqrt{y+2} + \sqrt{5y+1} = \sqrt{8y+7}$

20. $\sqrt{1-3z} = 2\sqrt{-1-2z} - \sqrt{-6-2z}$

21. $\dfrac{1}{x} - \dfrac{1}{a} - \dfrac{1}{b} = \dfrac{1}{x-a-b}$

22. $\dfrac{y^2}{6m^2} - \dfrac{5y}{6mr} - \dfrac{1}{r^2} = 0$

23. $(c-d)^2z^2 - (c^2-d^2)z + cd = 0$

24. $(a^2-4)x^2 - 2(a^2+4)x + a^2 - 4 = 0$

25. $\dfrac{y^2}{4k^2-h^2} = \dfrac{1}{4k^2-h^2} + \dfrac{y}{2kh}$

26. $\dfrac{1}{2x} + \dfrac{1}{2x-f} = \dfrac{1}{e} + \dfrac{1}{e-f}$

62. Equations of Quadratic Type

It happens sometimes that an equation may be transformed into a quadratic equation by replacing an expression involving the unknown by a single letter. Such equations are said to be of **quadratic type.**

Example 1. Solve the following equation of quadratic type:

$$(x^2-1)^2 - 5(x^2-1) + 6 = 0.$$

Solution: Let $x^2 - 1 = y$. Then the given equation becomes

$$y^2 - 5y + 6 = 0.$$

The new equation is easily solved by the method of factoring, and the roots are found to be $y = 2$ and $y = 3$.

Since $\qquad y = x^2 - 1,$

we have $\qquad x^2 - 1 = 2,$

which yields the roots $x = \pm\sqrt{3}$. *Ans.*

Also, $x^2 - 1 = 3,$

which gives the roots $x = \pm 2$. *Ans.*

EXAMPLE 2. Solve the equation

$$y^2 + 3y + \sqrt{y^2 + 3y - 2} = 22.$$

Solution: Let $\sqrt{y^2 + 3y - 2} = s.$

Then, $y^2 + 3y - 2 = s^2,$

and $y^2 + 3y = s^2 + 2.$

Thus, the given equation may be written

$$s^2 + 2 + s = 22,$$

or $$s^2 + s - 20 = 0.$$

The last equation has the roots $s = 4$ and $s = -5$. Hence, we have

$$\sqrt{y^2 + 3y - 2} = 4,$$

an equation which may be solved by first squaring both sides. We find

$$y = 3 \quad \text{or} \quad y = -6. \quad \textit{Ans.}$$

The check is left for the student. The second value of s results in the equation

$$\sqrt{y^2 + 3y - 2} = -5$$

which is discarded because -5 is not the principal square root of any number.

EXERCISES 50

Solve each of the following equations by the method of the preceding section. Use only principal roots where radicals or fractional exponents are involved.

1. $y^4 - 13y^2 + 36 = 0$
2. $x^4 - 29x^2 + 100 = 0$
3. $w^{-4} - 9w^{-2} + 20 = 0$
4. $z^{-6} + 63z^{-3} - 64 = 0$
5. $v^{-8} - 17v^{-4} + 16 = 0$
6. $x^{-3/2} - 26x^{-3/4} - 27 = 0$
7. $y^{1/3} - 1 - 12y^{-1/3} = 0$
8. $u^{1/4} + 2 - 8u^{-1/4} = 0$
9. $x + 7 - \sqrt{x + 7} - 2 = 0$
10. $\sqrt{y + 20} - 4\sqrt[4]{y + 20} + 3 = 0$

11. $z^2 + 3z + \sqrt{z^2 + 3z} - 6 = 0$

12. $2w^2 - 3w - 2\sqrt{2w^2 - 3w} - 3 = 0$

13. $2v^2 + v - 4\sqrt{2v^2 + v + 4} = 1$

14. $x^2 - 3x + 3\sqrt{x^2 - 3x + 5} = 13$

15. $(w^2 + 2w)^2 - 14(w^2 + 2w) - 15 = 0$

16. $(2y^2 - y)^2 - 16(2y^2 - y) + 60 = 0$

HINT: In Exercises 17 and 18, the first two terms are the first two terms of a perfect square.

17. $y^4 - 6y^3 + 7y^2 + 6y - 8 = 0$

18. $9z^4 - 6z^3 - 35z^2 + 12z + 20 = 0$

19. $\left(3x - \frac{2}{x}\right)^2 + 6\left(3x - \frac{2}{x}\right) + 5 = 0$

20. $v^2 - 12 + \frac{36}{v^2} - 4v + \frac{24}{v} = 5$

21. $\frac{1}{x - 4\sqrt{x}} = \frac{x - 4\sqrt{x}}{9}$

22. $\frac{18}{y - 5\sqrt{y}} = \frac{y - 5\sqrt{y}}{2}$

63. Applied Problems

The use of quadratic equations as a means for solving practical problems is illustrated by the following examples.

EXAMPLE 1. A group of students rented a cabin for \$160. When 2 of the group failed to pay their shares, the cost to each of the remaining students was \$4 more. How many students were in the group?

Solution: Let n = the number of students in the group.

Then, $n - 2$ = the number who paid for the cabin.

Also, $\frac{160}{n}$ = the number of dollars each student should have paid

$\frac{160}{n - 2}$ = the number of dollars each student except two actually paid.

By the conditions of the problem, we have

$$\frac{160}{n - 2} = \frac{160}{n} + 4.$$

This equation may be simplified to obtain

$$n^2 - 2n - 80 = 0;$$

and $$n = 10 \quad \text{or} \quad -8.$$

The positive root is found to check, and the negative one is of no significance in this problem. Hence, there were 10 students in the group. *Ans.*

Example 2. A manufacturer sells a certain article to a wholesaler for \$50, and the same article is retailed for \$100. If the retailer figures the same percentage gross profit on his selling price to the consumer as the wholesaler figures on his price to the retailer, what is the percentage gross profit for each?

Solution: Let x (expressed as a decimal fraction) = the percentage gross profit that the retailer makes.

Then, $100x$ = the number of dollars profit that the retailer makes, and $100 - 100x$ = the number of dollars that the retailer pays for the article.

Since the wholesaler's percentage profit is figured in the same manner,

$x(100 - 100x)$ = the number of dollars in the wholesaler's profit.

$(100 - 100x) - x(100 - 100x)$ = the number of dollars in the manufacturer's price.

Hence, $$(100 - 100x) - x(100 - 100x) = 50.$$

Simplification of the last equation gives

$$2x^2 - 4x + 1 = 0.$$

Hence, $$x = \frac{2 \pm \sqrt{2}}{2}$$

$$= 1.7071 \quad \text{or} \quad 0.2929.$$

However, by the conditions of the problem, x cannot be greater than unity; therefore, 29.29 per cent is the required percentage profit. *Ans.*

Example 3. In the design of a small machine, three wheels A, B, and C are to be mounted in the same plane, as shown in Figure 37. Wheel B

is to turn the other two wheels by means of the friction at the points of contact P and Q. The circumference of B is 12 in., and the circumference of A is to be 1 in. longer than that of C. It is further required that C make one more revolution than A while B makes one revolution. Find the diameter to be used for each of wheels A and C.

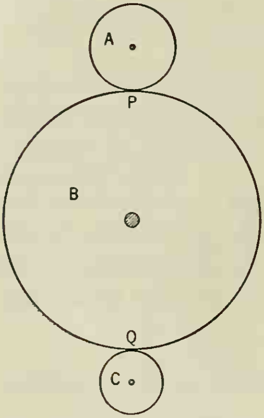


Fig. 37

Solution: Since the information in the problem concerns the circumferences of the wheels more directly than it does the diameters, we shall first find the unknown circumferences.

Let x = the number of inches in the circumference of C.

Then $x + 1$ = the number of inches in the circumference of A.

While B makes one revolution, a point on the circumference of either A or C will travel 12 in. Hence,

$\frac{12}{x}$ = the number of revolutions that C makes,

and $\frac{12}{x+1}$ = the number of revolutions that A makes while B makes one revolution.

As it is required that C make one more revolution than A in this time, we have

$$\frac{12}{x} = \frac{12}{x+1} + 1.$$

By multiplying both sides of this equation by $x(x + 1)$, we find

$$12x + 12 = 12x + x^2 + x,$$

or

$$x^2 + x - 12 = 0.$$

This equation has the two roots 3 and -4. Since the negative root has no significance in this problem, we have found that the circumference of C should be 3 in. The circumference of A should then be 4 in. The diameter of C is $3/\pi$, or 0.955 in. (approximately), and that of A is $4/\pi$, or 1.273 in. (approximately). *Ans.*

The check is left for the student.

EXERCISES 51

1. The sum of a number and its reciprocal is $2\frac{16}{21}$. Find the number.

2. Find two numbers whose difference is 20 and such that one of the numbers is the square of the other.

3. The altitude of a triangle is 10 in. shorter than the base, and the area is 252 sq in. Find the lengths of the base and altitude.

4. The area of a rectangle is 112 sq in. If one side is 6 in. shorter than the other, what are the dimensions of the rectangle?

5. Find the length of the side of a square whose diagonal is 4 in. longer than a side.

6. The upper base of a trapezoid is one third as long as its lower base, and its altitude is 3 ft shorter than its upper base. If the area of the trapezoid is 80 sq ft, find the altitude.

7. A rectangular box is 8 ft long, 4 ft wide, and 1 ft high. By what distance must the length be shortened to decrease the length of the body diagonal by 2 ft?

8. The hypotenuse of a right triangle is 5 ft longer than four times one of the legs. If the other leg is 40 ft long, find the length of the hypotenuse.

9. A 3 in. square is cut out of each corner of a rectangular piece of tin that is twice as long as it is wide. The ends are then turned up to make an open box whose volume is 324 cu in. What are the dimensions of the piece of tin?

10. A skating rink is 120 ft long and 80 ft wide. It is desired to triple the area by adding strips of equal width on one side and one end, and still keep the shape rectangular. Find the width of the strips.

11. A 60- by 100-ft building is surrounded by a walk of uniform width. Find the width of the walk if its area is 2064 sq ft.

12. When the radius of a spherical balloon is decreased by 6 in., the volume is decreased by 936π cu in. Find the original radius.

13. Two spheres whose volumes differ by 156π cu in. have radii that differ by 3 in. What is the radius of each?

14. A circle whose radius is 8 in. longer than the radius of a smaller circle has an area three times as great. Find the radius of each circle.

15. An open rectangular gutter is made by bending up the sides of a strip of metal 14 in. wide. What is the depth of the gutter if the carrying area is $22\frac{1}{2}$ sq in.?

16. A bathroom floor is covered with 972 square tiles. If each tile were $\frac{1}{2}$ in. longer on a side, it would take 768 tiles to cover the same area. Find the length of the side of one tile.

17. Two planes are flying at the rate of 300 mph on courses at right angles to each other. One plane is 54 miles from the crossing of their courses when the other is 62 miles from the same point. How many minutes later will they be 40 miles apart? How do you interpret the two different answers?

18. At 2 P.M. one ship is 8 miles south of another ship and is sailing south

at the rate of 24 mph. The second ship is sailing east at the rate of 15 mph. When will the ships be 26 miles apart?

19. A jet pilot, in flying 1200 miles, would have decreased his time by 25 min if he had flown 96 mph faster. Find his speed.

20. A person traveled 275 miles in an automobile. If his average speed had been 5 mph faster, his time for the trip would have been 30 min less. What was his average speed?

21. A group of students rented a bus for $80. If there had been 8 more students, the price per student would have been 50 cents less. How many students were in the group?

22. A certain large crew of men can do a job in 16 days less time than it takes another smaller crew. Together the two crews can do the job in 6 days. How long would it take each crew alone to do the job?

23. Two pipes running simultaneously can fill a tank in 20 min. The larger pipe, running alone, can fill the tank in 9 min less time than it would take the smaller pipe. How long would it take each pipe running separately to fill the tank?

24. A retailer paid $408 for a number of tires. When all but 10 tires had been sold at a profit of $5 each, the retailer had received the amount he paid for all of the tires. How many tires did the retailer buy?

25. A contractor agrees to do a certain construction job for $1200. It takes him 5 days less than he expected, and he therefore receives $12 more per day than he anticipated. In how many days did he expect to finish the construction?

26. A stone dropped from the top of a vertical cliff is heard striking the rocks at the base of the cliff 12 sec later. How high is the cliff? (Hint: Use the formula $s = 16t^2$ for the distance s in feet traveled by a falling body in t sec, and assume that sound travels 1100 ft per sec.)

27. A stone is dropped from the top of a building and is heard striking the pavement 6 sec later. Find the height of the building. (See the preceding problem.)

28. A lighthouse is 3 miles away from the nearest point P on a straight shore line. A man starts from point A on the shore line 16 miles from P and walks at the rate of 5 mph toward P. At a point B, between A and P, he boards a motor boat and goes straight to the lighthouse at the rate of 10 mph. If it takes 2 hr and 54 min to go from A to the lighthouse, locate the point B.

29. A large pole is supported by several guy wires, two of which are attached at a point 40 ft above the base. One of these wires is 8 ft longer than the other and is anchored to the ground at a point 12 ft farther from the base than the other. Find the length of each of these two wires.

Chapter 10

RATIO, PROPORTION, AND VARIATION

64. Ratio

The *ratio* of a number a to a number b is the indicated quotient $a \div b$, that is,

$$\textbf{The ratio of } a \textbf{ to } b = \frac{a}{b}.$$

Illustration: Let a line 12 units long be divided into two segments 7 units and 5 units long, respectively. Then the ratio of the first segment to the second segment is $\frac{7}{5}$; the ratio of the first segment to the whole line is $\frac{7}{12}$; and that of the second segment to the whole line is $\frac{5}{12}$.

EXAMPLE 1. If the ratio of a to b is $\frac{2}{3}$, find the ratio of $2a + 3b$ to $4a - b$.

Solution: We have given $\dfrac{a}{b} = \dfrac{2}{3}$, and are to find the value of

$$\frac{2a + 3b}{4a - b}.$$

If we divide the numerator and the denominator of the last fraction by b, we obtain

$$\frac{2\left(\dfrac{a}{b}\right) + 3}{4\left(\dfrac{a}{b}\right) - 1}.$$

Using the value given for $\frac{a}{b}$, we have

$$\frac{2\left(\frac{2}{3}\right) + 3}{4\left(\frac{2}{3}\right) - 1} = \frac{4 + 9}{8 - 3} = \frac{13}{5} \cdot \quad Ans.$$

Two physical quantities of the same nature are frequently compared by taking their ratio. Whenever this is done, the two quantities should be expressed in the same units.

Illustrations: (*a*) The ratio of 4 ft to 5 yd is $\frac{4}{15}$, since 5 yd are equivalent to 15 ft.

(*b*) The ratio of 2 hr to 15 sec is $\frac{7200}{15}$ or $\frac{480}{1}$ because 2 hr are equivalent to 7200 sec.

We sometimes wish to divide a number into three or more parts that bear given ratios to each other. Thus, the statement that a number is divided into three parts in the ratio $a : b : c$, read "a to b to c," means that the first and second parts have the ratio $\frac{a}{b}$, the second and the third parts $\frac{b}{c}$, and the first and third $\frac{a}{c}$. If the given number is N, and we let the three parts be ax, bx, and cx, where

$$ax + bx + cx = N,$$

we have

$$x = \frac{N}{a + b + c} \cdot$$

The three parts can now be found by multiplying this fraction by a, b, and c, respectively. It should be clear that this process yields the correct ratios.

Example 2. Divide \$135 into three parts in the ratio $3 : 5 : 7$.

Solution: From the preceding discussion, we see that 135 may be divided by the sum of the three members in the required ratio, and the result multiplied by each member, respectively, to obtain the required parts. Hence, we have

$$\frac{135}{3 + 5 + 7} = \frac{135}{15} = 9,$$

and $(3)(9) = 27,\ (5)(9) = 45,\ (7)(9) = 63,$

so that the required division is \$27, \$45, and \$63. *Ans.*

EXERCISES 52

Find the value of each ratio in Exercises 1 to 6.

1. 3 pt to 8 qt

2. 6 oz to 8 lb

3. 1 hr 5 min to 24 sec

4. 720 cu in. to 5 cu ft

5. 154 cu in. to 3 gal

6. 7 sq yd to 6 sq ft

7. Divide 636 into 3 parts in the ratio of 1 to 2 to 3.

8. Divide \$1800 into 3 amounts so that they are in the ratio of 3 to 5 to 7.

9. Find four numbers whose sum is 1280 and whose ratio is 2 : 4 : 5 : 9.

10. Divide \$1750 into four amounts in the ratio 4 : 7 : 11 : 13.

11. Find the ratio of $6a + b$ to $4a + 5b$ if $\frac{a}{b} = \frac{3}{4}$.

12. Find the ratio of $2c - 7d$ to $5c + 7d$ if $\frac{c}{d} = -2\frac{1}{3}$.

13. If $27x^2 + 24xy - 35y^2 = 0$, find the ratio of x to y.

14. If $12r^2 - 4rst - 33s^2t^2 = 0$, find the ratio of r to st.

15. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, show that $\frac{la + mc + ne}{lb + md + nf} = \frac{a}{b} = \frac{c}{d} = \frac{e}{f}$. (HINT: Since the ratios are all equal, let $\frac{la}{lb} = \frac{a}{b} = r$ and form equations such as $la = lbr$ for each ratio. Then add these equations.)

65. Proportion

An equation in which both members are ratios is called a **proportion.** Thus, the statement

$$\frac{a}{b} = \frac{c}{d}$$

is a proportion. This special type of equation is sometimes read "a is to b as c is to d."

The numbers a and d are the **extremes** and b and c the **means** in the proportion.

The number d is termed the **fourth proportional** to a, b, and c.

If the means of a proportion are equal, as in

$$\frac{a}{b} = \frac{b}{c},$$

the number c is termed the **third proportional** to a and b; and the number b is termed a **mean proportional** between a and c.

The next examples show how we may obtain simple properties of proportions by the direct algebraic manipulation of the equations involved.

EXAMPLE 1. Show that if four numbers are in proportion, the product of the means is equal to the product of the extremes.

Solution: If $\frac{a}{b} = \frac{c}{d}$, we may multiply both sides by bd to obtain

$$ad = bc.$$

EXAMPLE 2. Show that if the product of two numbers is equal to the product of two other numbers, either pair may be taken as the means, and the other pair as the extremes, of a proportion.

Solution: If $ad = bc$ is the given equation, we may divide both sides by bd to obtain

$$\frac{a}{b} = \frac{c}{d}.$$

Similarly,

$$\frac{a}{c} = \frac{b}{d};$$

$$\frac{d}{c} = \frac{b}{a};$$

and so forth.

EXAMPLE 3. If $\frac{a}{b} = \frac{c}{d}$, show that

$$\frac{a+b}{b} = \frac{c+d}{d}.$$

Solution: Since

$$\frac{a+b}{b} = \frac{a}{b} + 1,$$

and

$$\frac{c+d}{d} = \frac{c}{d} + 1,$$

we add 1 to both sides of the equation $\frac{a}{b} = \frac{c}{d}$ to get

$$\frac{a}{b} + 1 = \frac{c}{d} + 1,$$

or
$$\frac{a+b}{b} = \frac{c+d}{d}. \tag{1}$$

The last example is often stated: *If four quantities are in proportion, they are in proportion by addition.*

If 1 is subtracted from both sides of the equation $\frac{a}{b} = \frac{c}{d}$, then, in a manner similar to that in Example 3, we obtain

$$\frac{a-b}{b} = \frac{c-d}{d}. \tag{2}$$

The four numbers in the above equation are said to be in proportion *by subtraction.*

If we divide the members of

$$\frac{a+b}{b} = \frac{c+d}{d}$$

by the corresponding members of

$$\frac{a-b}{b} = \frac{c-d}{d},$$

we have
$$\frac{a+b}{a-b} = \frac{c+d}{c-d}. \tag{3}$$

The four quantities in the last equation are said to be in proportion *by addition and subtraction.*

The student may show conversely that the proportion $\frac{a}{b} = \frac{c}{d}$ may be derived from any one of Equations (1), (2), or (3).

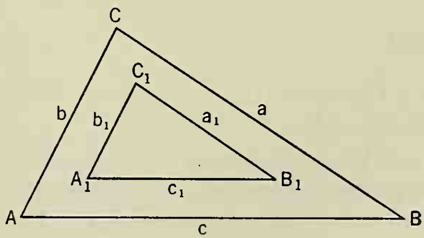


Fig. 38

Proportions are of considerable use in geometry in connection with *similar* figures, that is, roughly speaking, figures of the same shape. In particular, we say that two figures whose edges are straight lines are similar if their corresponding angles are equal and their corresponding sides are proportional. In Figure 38, *ABC* and

$A_1B_1C_1$ are two similar triangles. This means that

$$\text{angle } A = \text{angle } A_1,$$

$$\text{angle } B = \text{angle } B_1,$$

$$\text{angle } C = \text{angle } C_1,$$

and
$$\frac{a}{a_1} = \frac{b}{b_1} = \frac{c}{c_1}.$$

It is shown in geometry that if two triangles have their corresponding sides proportional, their corresponding angles are equal; that is, the triangles are similar, and conversely.

EXAMPLE 4. A triangle has sides a, 5 in. long; b, 7 in. long; and c, 8 in. long. A second triangle, which is similar to the first, has corresponding sides a_1, b_1, and c_1. If a_1 is 15 in. long, find the lengths of b_1 and c_1.

Solution: Since
$$\frac{a}{a_1} = \frac{b}{b_1},$$

we have
$$\frac{5}{15} = \frac{7}{b_1},$$

and
$$b_1 = 21.$$

In the same way,
$$\frac{a}{a_1} = \frac{c}{c_1},$$

or
$$\frac{5}{15} = \frac{8}{c_1},$$

and
$$c_1 = 24.$$

Therefore, the two sides b_1 and c_1 are 21 and 24 in. long, respectively. *Ans.*

The Pythagorean theorem, which states that the square of the hypotenuse of a right triangle equals the sum of the squares of the other two sides, may easily be derived by means of proportion. Thus, in Figure 39 let ABC be a right triangle with legs a and b and hypotenuse c. Let CP be the perpendicular dropped from C to the hypotenuse. Clearly, APC and BPC are right triangles. In the triangles ABC and APC, the angle A is a common angle, and the right angle at P is equal to the right angle at C. Consequently, the angles ACP and CBA are

equal, the triangles are similar, and the corresponding sides are proportional. Hence,

$$\frac{x}{b} = \frac{b}{c}. \tag{4}$$

In the same manner, triangles BPC and BCA may be used to find

$$\frac{c - x}{a} = \frac{a}{c}. \tag{5}$$

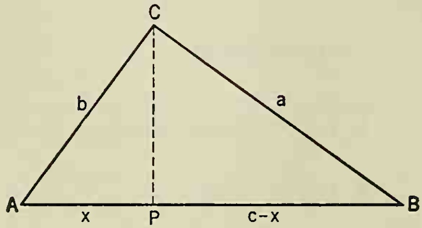


Fig. 39

Now, from Equation (4), we find

$$cx = b^2;$$

and, from Equation (5),

$$c^2 - cx = a^2,$$

or
$$c^2 = a^2 + cx.$$

By combining these results, we obtain the Pythagorean theorem, namely,

$$c^2 = a^2 + b^2.$$

For two similar triangles with bases b_1 and b_2, respectively, and corresponding altitudes h_1 and h_2, we have for the areas

$$A_1 = \tfrac{1}{2}b_1h_1 \quad \text{and} \quad A_2 = \tfrac{1}{2}b_2h_2;$$

hence,
$$\frac{A_1}{A_2} = \frac{b_1h_1}{b_2h_2}.$$

Since
$$\frac{h_1}{h_2} = \frac{b_1}{b_2},$$

we find
$$\frac{A_1}{A_2} = \frac{b_1^2}{b_2^2}.$$

This result is a special case of the important rule that *corresponding areas of similar figures are proportional to the squares of corresponding lines.*

It is further shown in geometry that *the volumes of similar solids are proportional to the cubes of corresponding lines.*

Example 5. What is the ratio of the areas of the bases of two similar pyramids of altitudes 2 in. and 3 in., respectively?

Solution: From the preceding discussion, we find that the areas of the bases are proportional to the squares of the altitudes of the two

pyramids, that is,

$$\frac{B_1}{B_2} = \frac{h_1^2}{h_2^2}.$$

Consequently, the given values of h_1 and h_2 result in

$$\frac{B_1}{B_2} = \frac{2^2}{3^2} = \frac{4}{9}. \quad \textit{Ans.}$$

EXERCISES 53

1. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a + 4b}{a - 4b} = \frac{c + 4d}{c - 4d}$.

2. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{5a + 3b}{2a - 7b} = \frac{5c + 3d}{2c - 7d}$.

3. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a^2 - c^2}{a^2 + 3c^2} = \frac{b^2 - d^2}{b^2 + 3d^2}$.

4. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{3ab + 2cd}{ab - cd} = \frac{3b^2 + 2d^2}{b^2 - d^2}$.

5. If $\frac{a}{b} = \frac{b}{c}$, show that $\frac{a^2}{b^2} = \frac{6a^2 + 11b^2}{6b^2 + 11c^2}$.

6. Find the fourth proportional to (*a*) 8, 12, and 18; (*b*) 0.3, 0.4, and 0.75; (*c*) $x^2 - y^2$, $x - y$, and $mx + my$; (*d*) r, s, and t.

7. Find the mean proportionals between the two numbers in each of the following pairs: (*a*) 75 and 5; (*b*) 63 and 28; (*c*) $e - f$ and $e^3 - f^3$; (*d*) $25c + 25d$ and $4c^2 - 4d^2$.

8. Find the third proportional to (*a*) 75 and 15; (*b*) 0.08 and 0.24; (*c*) a^2 and b^2; (*d*) $9e^5$ and $18\ ef^2$.

9. The sides of a triangle are 15, 18, and 30 units long, respectively. If the shortest side of a similar triangle is 10 units long, what are the lengths of the other sides?

10. Two triangles with sides a_1, b_1, c_1 and a_2, b_2, c_2, respectively, are similar. Show that a triangle with sides $a_1 + 2a_2$, $b_1 + 2b_2$, $c_1 + 2c_2$ is similar to the given triangles.

11. A man 6 ft tall is 20 ft from the point directly beneath a street light. What is the length of his shadow if the light is 34 ft above the street?

12. Two similar polygons have sides which are in the ratio of 3 to 2. If the area of the smaller polygon is 60 sq in., what is the area of the larger polygon?

13. If the surface areas of two spheres are s_1 and s_2, the radii r_1 and r_2, and the diameters d_1 and d_2, respectively, show that $\frac{s_1}{s_2} = \frac{r_1^2}{r_2^2} = \frac{d_1^2}{d_2^2}$.

14. Compare the surface areas of two spheres whose radii have the ratio (*a*) $\frac{5}{2}$; (*b*) $\frac{3}{7}$.

15. Compare the radii of two spheres whose surface areas are in the ratio (*a*) 25 : 64; (*b*) 81 : 121.

16. The diameter of the planet Jupiter is 11 times that of the earth. How many times larger is the surface of Jupiter than that of the earth?

17. Show that for two spheres $\frac{V_1}{V_2} = \frac{r_1^3}{r_2^3} = \frac{d_1^3}{d_2^3}$, where V stands for volume, r for radius, and d for diameter.

18. Compare the volumes of two spheres whose radii have the ratio (*a*) $\frac{2}{9}$; (*b*) $\frac{8}{5}$.

19. Compare the radii of two spheres whose volumes are in the ratio (*a*) 1 : 8; (*b*) 125 : 216.

20. The diameter of the sun is approximately 109 times that of the earth. Compare the surface areas and the volumes of the two bodies.

66. Variation

Suppose that a coil spring hangs in a vertical position from a support at its upper end. A weight, attached to the lower end and allowed to fall slowly to a position of equilibrium, will cause the spring to extend, the length of the extension depending on the magnitude of the weight. Within certain limits, it is found that W, the number of units in the magnitude of the weight, is proportional to s, the number of units in the length of the corresponding extension. Thus, if W_1 and W_2 are two different values of W, and s_1 and s_2 are the corresponding values of s, then

$$\frac{W_1}{s_1} = \frac{W_2}{s_2}.$$

Hence, the ratio of any value of W to the corresponding value of s is a constant, say k, which is called the *modulus* of the spring. Within the limits mentioned above, we may write

$$W = ks.$$

If the value of k for a given spring is known, the weight needed for a given extension may be found. For example, if k is 25 lb per in., we have

$$W = 25s,$$

a formula for W in pounds in terms of s in inches.

There are many places in science and technology where the student will encounter ideas similar to those in the preceding paragraph. Since it is usual to state these ideas in the language of *variation*, we shall explain and use some of the more common phrases.

1. *Direct Variation.* The statements

y *varies as* x,

y *varies directly as* x,

and y *is proportional to* x

are all equivalent. Each is translated into the equation

$$y = kx,$$

where k is a ***constant of proportionality.***

Since $y = kx$ is equivalent to

$$\frac{y}{x} = k, \qquad (x \neq 0)$$

it is clear that the constant of proportionality may be found if a pair of corresponding values (x_1, y_1), where $x_1 \neq 0$, is given. Furthermore, if (x_1, y_1) and (x_2, y_2) are two pairs of corresponding values, we have

$$\frac{y_1}{x_1} = \frac{y_2}{x_2}.$$

Illustration: If a gas such as air is enclosed in a container of fixed volume, the pressure of the gas on the walls of the container varies directly as the absolute temperature of the gas. If P is the pressure in pounds per square inch and T is the temperature in degrees Fahrenheit absolute, then

$$P = kT.$$

(*Absolute* temperature is measured from the so-called *absolute zero.* The zero point is approximately 460°F below 0°F or 273°C below 0°C.)

NOTE: The looser statement "y varies *with* x" means only that y is a function of x; the kind of function is not specified.

EXAMPLE 1. The air in a closed cylinder exerts a pressure of 5 lb per sq in. when the absolute temperature is 460°F. Find the value of the constant of proportionality. What will be the pressure if the temperature is raised to 560°F absolute?

Solution: From the illustration above, we have

$$P = kT.$$

Since $P = 5$ when $T = 460$, we must have

$$5 = k(460),$$

or $$k = \frac{5}{460} = \frac{1}{92}.$$

For this particular situation, the relation between P and T is

$$P = \frac{1}{92}T.$$

If $T = 560$, $$P = \frac{560}{92} = 6.1 \quad \text{approximately.}$$

Hence, the pressure will be 6.1 lb per sq in. at a temperature of 560°F absolute. *Ans.*

When the value of k is not required, it may be easier to work as follows: If P is taken as the unknown pressure, T the corresponding temperature, and P_1 and T_1 as the given pressure and temperature, respectively, then

$$\frac{P}{P_1} = \frac{T}{T_1},$$

that is, $$\frac{P}{5} = \frac{560}{460} = \frac{28}{23};$$

so $$P = \frac{140}{23} = 6.1 \quad \text{approximately.}$$

2. *Inverse Variation.* The statements

y *varies inversely as* x,

and

y *is inversely proportional to* x

are equivalent; they are translated into the algebraic statement,

$$y = \frac{k}{x},$$

where k is a constant.

If (x_1, y_1) and (x_2, y_2) are two pairs of corresponding values, we have

$$y_1 = \frac{k}{x_1} \quad \text{or} \quad x_1y_1 = k,$$

and $$y_2 = \frac{k}{x_2} \quad \text{or} \quad x_2y_2 = k.$$

Hence, $$x_1y_1 = x_2y_2.$$

Illustration: If the temperature of a gas such as air is held constant, its pressure varies inversely as its volume. If P is the pressure and V the corresponding volume, we have

$$P = \frac{k}{V},$$

or

$$PV = k. \qquad \text{(This is Boyle's Law.)}$$

Example 2. A pressure of 1760 lb per sq ft is exerted by 2 cu ft of air in a cylinder fitted with a piston. If the piston is pushed out until the pressure becomes 1320 lb per sq ft, what will be the volume of air if there is no change in temperature? Find the value of the constant of proportionality.

Solution: From the preceding illustration, we have

$$PV = k.$$

Hence,

$$k = (2)(1760) = 3520. \quad \textit{Ans.}$$

(P must be in pounds per square foot and V in cubic feet for this value of k.)

Thus, the general relation between P and V is

$$PV = 3520.$$

If $P = 1320$,

$$V = \frac{3520}{P} = \frac{3520}{1320} = \frac{8}{3} = 2\tfrac{2}{3}.$$

That is, the volume will be $2\frac{2}{3}$ cu ft when the pressure is reduced to 1320 lb per sq ft. *Ans.*

3. *Joint Variation.* The statements

z varies jointly as x and y

and

z is proportional to x and y

are equivalent. The translation of either statement into algebraic symbolism is

$$z = kxy,$$

where k is a constant of proportionality.

Remark: The student must observe the warning that the word "and" in the language of variation usually indicates a multiplication as in joint variation.

Example 3. The lifting force P in pounds exerted by the atmosphere on the wings of an airplane varies jointly as the area A of the wings in

square feet and the square of the plane's speed V in miles per hour. If the lift is 1200 lb for a wing area of 100 sq ft and a speed of 75 mph, find the formula for P in terms of A and V.

Solution: The statement of the problem is translated

$$P = kAV^2.$$

Hence,
$$1200 = k(100)(75)^2;$$

that is,
$$k = \frac{4}{1875}.$$

Thus, the desired formula is

$$P = \frac{4AV^2}{1875}.$$

Notice in problems such as the last one that the constant of proportionality is determined for the particular set of units stated in the problem. For example, from the formula $P = kAV^2$, in Example 3, we have

$$k = \frac{P}{AV^2}.$$

Since P is in pounds, A in square feet, and V in miles per hour, k is measured in the units

$$\frac{\text{lb}}{(\text{ft}^2)(\text{mi/hr})^2}.$$

In general, the value of k will depend on the set of units used for the other quantities; hence, the student should always make certain that the units he uses are consistent with those for which the formula is derived.

Example 4. Newton's law of gravitational attraction states that the force F with which two particles of mass m_1 and m_2, respectively, attract each other varies directly as the product of the masses and inversely as the square of the distance r between them. Write this in the form of an equation. If one of the masses is doubled and the distance between the masses is also doubled, what happens to the force?

Solution:
$$F = k\frac{m_1m_2}{r^2}. \quad \textit{Ans.}$$

Let F_1 be the force when the mass m_1 is doubled and the distance r is

also doubled. Then

$$F_1 = k\frac{(2m_1)m_2}{(2r)^2} = k\frac{m_1m_2}{2r^2}.$$

By comparison with the previous force F, we see that

$$\frac{F_1}{F} = \frac{k\dfrac{m_1m_2}{2r^2}}{k\dfrac{m_1m_2}{r^2}} = \frac{1}{2}.$$

That is, the force is halved if one mass and the distance between the masses are both doubled. *Ans.*

EXERCISES 54

1. If z is proportional to v^2 and $z = 80$ when $v = 4$, find z when $v = 9$.

2. If y varies inversely as x^2 and $y = 72$ when $x = 6$, find y when $x = 12$.

3. If w is proportional to the product of u^2 and t^3, and $w = 384$ when $u = 4$ and $t = 2$, find w when $u = 6$ and $t = 3$.

4. If z varies directly as x^3 and inversely as $y^{1/2}$, and $z = 108$ when $x = 6$ and $y = 8$, find z when $x = 3$ and $y = 18$.

5. If p varies directly as s^2, and if s varies directly as t^3, show that p varies directly as t^6.

6. If u varies inversely as v^2, and if v varies inversely as w^2, show that u varies directly as w^4.

7. The kinetic energy of a moving body is proportional to the square of its velocity. Compare the kinetic energy of a skier when he skis at 15 mph with his kinetic energy when he skis at 60 mph.

8. The weight of a model dam varies directly as the cube of its height. What would be the weight of a model dam 64 in. high if a similar model 12 in. high weighs 243 lb?

9. The amount of sediment which a stream will carry is directly proportional to the sixth power of its speed. How much more sediment will a stream carry if its velocity is increased fifty per cent?

10. If the temperature of a gas remains constant, the pressure P varies inversely as the volume V. A gas at a pressure of 16 lb per sq in. and having a volume of 500 cu ft is compressed to a volume of 25 cu ft. What is the final pressure if the temperature remains constant?

11. The wind force F on a vertical surface varies jointly as the area A of the surface and as the square of the wind velocity V. When the wind is blowing 20 mph, the force on 1 sq ft of surface is 1.8 lb. Find the force exerted on a surface 2 ft square when the wind velocity is 60 mph.

12. The horsepower that a rotating shaft can safely transmit varies jointly

as the cube of its diameter and the number of revolutions it makes per minute. If a 2-in. shaft rotating at 1400 rpm can safely transmit 500 hp, what horsepower can a 4-in. shaft transmit at 2100 rpm?

13. A group of thirty-five men can do a certain piece of work in fifteen 8-hr days. How many men can do the same work in twenty-four 7-hr days?

14. On a certain truck line, it costs $9.80 to send 5 tons 7 miles. How much will it cost to send 12 tons 18 miles?

15. The force of attraction between two spheres varies as the product of their masses and inversely as the square of the distance between their centers. If, for a given distance, the force of attraction between two spheres is 360 dynes, what is the attractive force when the distance is tripled? What is the attractive force when each mass is quadrupled and the distance is doubled?

16. The force with which the earth attracts an object above the earth's surface varies inversely as the square of the distance from the center of the earth. How much will a 72-lb meteorite weigh when it is 1980 miles above the earth's surface? How far above the earth's surface will it be when it weighs 18 lb? (The radius of the earth is approximately 3960 miles.)

17. The distance d in miles that a person can see to the horizon from a point h ft above the surface of the earth varies approximately as the square root of the height h. If for a height of 600 ft the horizon is 30 miles distant, how far is the horizon from a point which is 1176 ft high?

18. If in the preceding problem, the original height is nine times as great, how much farther can the observer see?

19. The f numbers on a camera lens and shutter vary inversely as the diameter of the aperture, provided that the distance setting is infinity. If, for a given lens, the aperture diameter is $\frac{1}{4}$ in. when the f number is 8, what is the f number for a diameter of $\frac{1}{8}$ in.? for a diameter of 2 in.?

20. The time of exposure necessary to photograph an object varies as the square of the distance d of the object from the light source and inversely as the intensity of illumination I. For a given setup the correct exposure is $\frac{1}{50}$ sec when the light is 5 ft from the object. What must be the distance of the object from the light source if the light intensity is doubled and the exposure time is increased to $\frac{1}{10}$ sec?

21. The pressure P of a gas varies directly as its absolute temperature T and inversely as its volume V. A gas whose volume is 600 cu ft and whose temperature is 47°C has a pressure of 150 lb per sq in. The gas is allowed to expand until its pressure is 15 lb per sq in. and its temperature is 7°C. What is its final volume?

22. The volume and the absolute temperature of a gas are each trebled in value. What change is there in the pressure? If the original pressure is doubled, and the absolute temperature is decreased to one half of its original value, what is the ratio of the final volume to the original volume?

23. The quantity of water Q that flows from an orifice in a given time is proportional to the area A of the orifice and the square root of the depth h of the orifice below the surface. There are two orifices in the side of a reservoir which is kept at a constant level. The ratio of their areas is 3.5 : 1 and the ratio of their depths is 2.25 : 1. Compare the quantity of water which passes

through the larger orifice with that which passes through the smaller orifice in a given time.

24. Kepler's third law states that the square of the time it takes a planet to make one circuit about the sun varies as the cube of its mean distance from the sun. The mean distance of the earth from the sun is 92.9 million miles, and the mean distance of Venus from the sun is 67.2 million miles. Find the time it takes Venus to make one circuit about the sun.

25. The illumination I in foot-candles upon a wall varies directly as the intensity i in candlepower of the source of light and inversely as the square of the distance d from the light. If the illumination is 5 ft-c at a distance of 10 ft from a light of 500 cp, what is the illumination at a distance of 15 ft from a light of 3000 cp?

26. How many times as great is the illumination on an object 5 ft distant from a 200 cp light as it is upon an object 20 ft distant from a 10-cp light?

27. The strength S of a horizontal beam of rectangular cross section and of length L when supported at both ends varies jointly as the breadth b and the square of the depth d and inversely as the length L. A 2- by 4-in. beam 8 ft long and resting on the 2-in. side will safely support 600 lb. What is the safe load when the beam is resting on the 4-in. side?

28. A 4- by 6-in. beam 12 ft long and resting on edge will support 4000 lb. What weight would be supported by a 6- by 10-in. beam 16 ft long and resting on edge? Assume that the second beam is made of the same material as the first. (See the preceding problem.)

29. The electrical resistance of a wire of uniform cross section varies directly as its length and inversely as its cross-sectional area. A solid cube of copper 1 in. on a side has a resistance of 6.85×10^{-7} ohms at 20°C. At the same temperature, what is the resistance of a copper wire 200 ft long and 0.02 in. in diameter?

30. What is the diameter of a copper wire 1000 ft long whose resistance is 2 ohms?

31. If the resistance of a wire 700 ft long and 0.01 in. in diameter is 20 ohms, what is the resistance of a wire of the same material 2100 ft long and 0.04 in. in diameter?

32. A piece of aluminum wire is six times as long as another piece of aluminum wire, and its radius is one fifth as large. The resistance of the first wire is how many times as great as the resistance of the second wire?

33. The cost of gasoline per hour for running a plane is proportional to the square of the speed and is $80 per hour when the speed is 200 mph. Other charges per hour total $50 regardless of speed. Find the expression for the cost per mile as a function of the speed v.

Chapter 11

APPROXIMATE NUMBERS

67. Approximate Numbers

In making scientific computations, the computer must always be on guard against the tendency to present results with an appearance of accuracy that they actually do not possess. The possibility of obtaining such false accuracy is always present when the data in the problem involve approximate numbers.

A number that is the direct result of a measurement invariably represents the measured quantity only approximately. For instance, suppose we measure the length and width of a rectangular table top, *correct to the nearest tenth of an inch*, and we find the measurements to be 50.2 in. and 29.6 in., respectively. All that we can say with assurance is that the actual length of the table is closer to 50.2 in. than it is to 50.1 or to 50.3 in.; so the dimension is somewhere between 50.15 and 50.25 in. That is what we mean by "correct to the nearest tenth of an inch." Similarly, the width is somewhere between 29.55 and 29.65 in. Anyone who objects to this lack of precision has only one recourse—to measure the distances more accurately. If we obtain a better measurement of the length, as 50.23 in., correct to the nearest hundredth of an inch, we know that the actual length lies somewhere between 50.225 and 50.235 in. Although we may narrow the interval in which the length must lie, we still do not know the "true" length. Our knowledge of this distance is limited by the precision of our measuring instruments; and, no matter how precise these are, we can never know beyond all doubt the exact value of the measured quantity.

68. Addition and Subtraction of Approximate Numbers

The last remarks lead at once to very important qualifications to be placed on quantities computed from approximate data. For example, let it be required to find the perimeter of the table top of the preceding discussion. Suppose that the best value known for the length is 50.23 in., and for the width is 29.6 in. Twice the sum of these two distances is 159.66 in. However, the hundredth's digit in the width is unknown; consequently, the corresponding digit in the answer is of little value. The result should therefore be rounded off to 159.7 in.; even now, the last digit 7 is of doubtful accuracy. (The student should review the rules for rounding off given in Section 41.)

In the same manner, the difference between the length and the width should be written

$$50.23 - 29.6 = 20.6 \text{ in.}$$

The digit 3, which is obtained by actual subtraction, is discarded.

The preceding discussion contains the essential ideas of the general procedure which leads to the rule: *The result of an addition or subtraction of approximate numbers must never be given to more decimals than is possessed by that one of the numbers with the fewest decimal places.* We usually give the result to the largest number of decimal places that is compatible with the rule.

EXAMPLE 1. Add the approximate numbers 21.262, 23.75, and 39.6.

Solution:

$$\begin{array}{l} 21.262 \\ 23.75 \\ \underline{39.6} \\ 84.612 \end{array}$$

The answer, correctly rounded off, is 84.6.

NOTE: Some computers retain all the given digits in the intermediate work; others round off the data to one or two more places than are to be retained in the answer. In either case, the final result must be rounded off according to the preceding rule.

EXAMPLE 2. Subtract the third number in Example 1 from the sum of the other two.

Solution:

$$\begin{array}{r} 21.262 \\ \underline{+23.75} \\ 45.012 \\ \underline{-39.6} \\ 5.412 \end{array}$$

The answer, correctly rounded off, is 5.4.

EXERCISES 55

Each number in these exercises is approximate and is to be taken as correct only to the stated number of decimal places; answers are to be rounded off accordingly.

Add the numbers in each of the first eight examples.

1.	**2.**	**3.**	**4.**
6.09	92.33	1.962	0.1562
5.4	4.785	0.3427	0.87
849.	3.5	2.441	0.591
7.01	7.13	3.29	2.81

5.	**6.**	**7.**	**8.**
93.57	37.53	0.003	4.24
9.132	6.81	1.4729	197.3
70.00	14.9	0.051	2.148
87.4	53.0	0.126	61.42

Subtract the lower number from the upper number in each of the following:

9.	**10.**	**11.**	**12.**
54.17	7.314	295.36	57.7
6.3	2.53	52.8	1.36

13.	**14.**	**15.**	**16.**
26.85	0.081	4.317	0.3275
3.726	0.0969	0.120	0.68

69. Significant Digits; Multiplication and Division

Before the rule which covers multiplication and division can be stated, it is necessary to define *significant digits.*

(1) *The digits 1, 2, 3, 4, 5, 6, 7, 8, and 9 are always significant.*

(2) *The digit 0 is significant if it is preceded and followed by other significant digits.*

(3) *The digit 0 is never significant when its only function is to place the decimal point.*

Illustrations: All five digits in each of the numbers 20,675 and 3.0069 are significant.

The number 0.00261 has only three significant digits: 2, 6, and 1; the zeros in this number have no function except to place the decimal point.

It is generally agreed that, when an approximate number has a decimal part, *final* zeros are to be regarded as significant. For example, a measurement given as 5.60 ft is to be considered to have three significant digits. Notice that the position of the decimal point has nothing to do with the number of significant digits.

There is one ambiguous case left; for example, how many significant

digits does the number 53,200 have? This question cannot be answered without further information about the nature of the final zeros. In order to avoid this difficulty, we use the procedure of writing a number as *a number between 1 and 10, in which all significant digits are displayed, multiplied by a power of 10.* Thus, we would write 53,200 in the form

5.32×10^4, if only three digits are significant;

5.320×10^4, if only four digits are significant;

5.3200×10^4, if all five digits are significant.

We shall call this way of writing numbers **standard notation**; in standard notation, the decimal point is invariably placed immediately after the first digit on the left. The power of 10 which is to be used can always be written by inspection. Since a multiplication by 10 moves the decimal point one place to the right and a division by 10 moves the point one place to the left, *the exponent of 10 is always the number of places the decimal point is displaced from standard position in the given number. The sign of the exponent is positive if the point is to the right of standard position, and negative if it is to the left of standard position.*

Illustrations:

$$67932 = 6.7932 \times 10^4;$$

$$10.237 = 1.0237 \times 10^1;$$

$$0.00500 = 5.00 \times 10^{-3}.$$

Remarks: A rule equivalent to the one just given for the determination of the exponent of 10 is as follows:

(*a*) *If the given number has an integral part, the exponent of 10 is positive or zero and is equal to one less than the number of digits to the left of the decimal point.*

(*b*) *If the given number has no integral part, the exponent is negative and is numerically equal to one more than the number of zeros between the decimal point and the first nonzero digit.*

Thus, in the preceding illustrations, the exponent 4 corresponds to 5 digits to the left of the point; the exponent 1 corresponds to 2 digits to the left of the point; the exponent -3 corresponds to 2 zeros between the point and the digit 5.

One value of this type of notation should now be clear inasmuch as the significant digits in any approximate number are exactly those used to the left of the multiplication sign when the number is written in standard notation.

(This notation is also called **scientific notation.** It is frequently used for writing very large or very small numbers. For example, the nucleus of an atom has an estimated diameter of less than 3×10^{-12} cm; there are approximately 6.02×10^{23} molecules in 22.4 liters of any gas at 0°C and under a pressure of 76 cm of mercury.)

Let us now continue to a consideration of the multiplication and division of approximate numbers. Suppose that it is required to use the measurements 50.2 in. and 29.6 in. to find the area of the table top of Section 67. We find that

$$(50.2)\ (29.6) = 1485.92.$$

How much of this result is dependable? We may answer this question in the following way: 50.2 is a measurement of some number between 50.15 and 50.25; 29.6 represents in the same way a number between 29.55 and 29.65. Hence, the required number must lie between

$$(50.15)\ (29.55) = 1481.9325$$

and

$$(50.25)\ (29.65) = 1489.9125.$$

The first three digits, 1, 4, and 8, of these two products agree. If we round off our original result to its first three digits, we get 1490, which, by comparison with the last two results, has a possible error of not more than one unit in the digit 9. Hence, we have sufficient justification for presenting our result as 1.49×10^3 sq. in. Notice that we had three significant digits given in each dimension, 50.2 in. and 29.6 in., and we found that the first three digits of the result 1490 were reasonably dependable.

The student should try examples in division as well as other examples in multiplication of approximate numbers to see that the following rule gives results that are fairly satisfactory: *The result of a multiplication or a division of approximate numbers must never be given with more significant digits than are possessed by that one of the numbers which has the fewest significant digits.*

We generally state the result with as many digits as possible without violating the rule. However, a detailed analysis such as that in the preceding discussion may be made whenever it is desirable.

Example 1. Find the product of the approximate numbers 0.00227 and 35.63.

Solution: $(0.00227)\ (35.63) = 0.0809.$ *Ans.*

This answer is rounded off to three significant digits from 0.0808801.

Notice that the position of the decimal point is not involved in determining the number of significant digits in the result.

EXAMPLE 2. Find the diameter of a circle whose circumference is measured as 6.2 ft.

Solution: The formula for the circumference C of a circle of diameter D is

$$C = \pi D.$$

Hence,
$$D = \frac{C}{\pi} = \frac{6.2}{3.14} = 2.0 \text{ ft.} \quad \textit{Ans.}$$

This answer is rounded off from 1.97. It should be observed that the approximation of π was taken to three significant digits, one more than it was proposed to keep in the final result.

NOTE: In calculations, exact numbers have no influence on the accuracy of the result. Numbers whose decimal approximations are read from tables, or numbers such as π, should be read with enough significant digits so that the accuracy of the given data is not invalidated. In the intermediate calculations, at least one extra digit should be carried whenever possible.

EXERCISES 56

Write each of the following numbers in standard notation. Assume that no final zeros are significant.

1. 600,000	**2.** 12,000	**3.** 56,710,000	**4.** 3,825,000
5. 7293	**6.** 824,500	**7.** 64,000	**8.** 218,350
9. 0.243	**10.** 0.00157	**11.** 0.0592	**12.** 0.000458
13. 0.03648	**14.** 0.000082	**15.** 0.0408	**16.** 0.0000007

Assuming that all final zeros are significant, perform the following multiplications and divisions and write the answers in standard notation:

17. (2,000)(8.00)
18. (0.120)(6.00)
19. (700.0)(0.40)
20. (4.00)(0.030)
21. (456)(0.00500)
22. (3413)(0.0002)
23. (7.6)(2.95)
24. (8.91)(3.72)
25. (5.68)(0.1527)
26. (0.12)(0.3)(649)
27. (3.6)(8.3)(42.4)
28. (5.60)(12.88)(6.032)
29. 90,000 ÷ 3.0
30. 0.00700 ÷ 20.0
31. 0.0420 ÷ 0.14
32. 4800 ÷ 0.12
33. 48.0 ÷ 1.20
34. 48.00 ÷ 0.1200
35. 3.95 ÷ 2.24
36. 82.4 ÷ 2500
37. 0.04562 ÷ 0.000240

In each of the following problems, assume that the data are correct only to the stated number of significant digits:

38. Find the area of a triangle whose altitude is 7.8 in. and whose base is 13.5 in.

39. Compute the area of a trapezoid whose altitude is 19.8 mm and whose bases are 3.4 mm and 34.2 mm, respectively.

40. Calculate the volume and the surface area of a cube 9.6 ft on an edge.

41. What is the area of a circle whose radius is 4.54 in.?

42. Find the volume and the surface area of a sphere whose radius is 6.72 cm.

43. Find the altitude of a triangle whose base is 32.6 in. and whose area is 834 sq in.

44. A glass jar holds 596.7 cu in. of liquid. What is the capacity of the jar in gallons? (By law, a gallon is equivalent to 231 cu in.)

45. If the circumference of a circle is 45.20 ft, what is its radius?

46. A plane flies for 1 hr 24 min 36 sec at an average speed of 312 mph. What distance does the plane fly? (Assume the time to be correct to the nearest second.)

47. What is the average speed of a plane which flies 946 miles in 2.8 hr? If the timing were required to be correct to three significant figures, by how many seconds could it vary?

Chapter 12

LOGARITHMS

70. Logarithms

An examination of the following table of powers of 2 reveals certain facts that are of particular importance to us in this chapter:

$$2^1 = 2 \qquad 2^6 = 64$$
$$2^2 = 4 \qquad 2^7 = 128$$
$$2^3 = 8 \qquad 2^8 = 256$$
$$2^4 = 16 \qquad 2^9 = 512$$
$$2^5 = 32 \qquad 2^{10} = 1024$$

If we use the base 2, the exponent 2 corresponds to the number 4, the exponent 3 corresponds to the number 8, and so on.

Furthermore,

$$(8)(32) = (2^3)(2^5) = 2^{3+5} = 2^8,$$
$$(8)(32) = 256,$$

and

$$2^8 = 256.$$

Again,

$$(4)(256) = (2^2)(2^8) = 2^{2+8} = 2^{10},$$
$$(4)(256) = 1024,$$

and

$$2^{10} = 1024.$$

Thus, the addition of the exponents 3 and 5 corresponds to the multiplication of the numbers 8 and 32; and the addition of the exponents 2 and 8 corresponds to the multiplication of the numbers 4 and 256. Similarly,

$$1024 \div 16 = 2^{10} \div 2^4 = 2^{10-4} = 2^6,$$
$$1024 \div 16 = 64,$$

and

$$2^6 = 64.$$

Hence, the subtraction of the exponent 4 from the exponent 10 corresponds to the division of 1024 by 16.

These correspondences suggest that we replace the operations of multiplication and division by the addition and subtraction of exponents, a procedure that is extremely useful in many arithmetical computations as well as in theoretical demonstrations.

In order to carry out this idea it is necessary to have a more convenient terminology than we have previously used. We substitute the name **logarithm** for exponent and say, for example, *the logarithm of 8 to the base 2 is 3,* in place of "if we use the base 2, the exponent 3 corresponds to the number 8." Similarly, *the logarithm of 27 to the base 9 is 1.5* means "if 9 is used as the base, the exponent which corresponds to 27 is 1.5." More generally, we use the following:

Definition: The logarithm of a number N *to the base* b *is the exponent which must be applied to* b *to give* N. The abbreviation $\mathbf{log}_b$ is used for "logarithm to the base *b*." Thus, the equation

$$log_b N = x$$

means $$N = b^x.$$

Illustrations: (*a*) Since $2^5 = 32$, $\log_2 32 = 5$.
(*b*) Since $27^{4/3} = 81$, $\log_{27} 81 = \frac{4}{3}$.
(*c*) Since $10^{-2} = 0.01$, $\log_{10} 0.01 = -2$.

Example 1. Write the equation $5^3 = 125$ in logarithmic form.

Solution: $$\log_5 125 = 3. \quad \textit{Ans.}$$

Example 2. Write the equation $\log_{32} 64 = \frac{6}{5}$ in exponential form.

Solution: $$(32)^{6/5} = 64. \quad \textit{Ans.}$$

Example 3. What is the logarithm of 1,000,000 to the base 10?

Solution: $$\text{Since } 10^6 = 1{,}000{,}000,$$

$$\log_{10} 1{,}000{,}000 = 6. \quad \textit{Ans.}$$

Example 4. What number has the logarithm $\frac{4}{3}$ if the base is 8?

Solution: This question is equivalent to finding x from the equation

$$8^{4/3} = x.$$

Hence, we have $$x = 16. \quad \textit{Ans.}$$

EXAMPLE 5. What base is used if $\log_b 216 = \frac{3}{2}$?

Solution: This is equivalent to finding b from the equation

$$b^{3/2} = 216.$$

Therefore, $$b = (216)^{2/3} = 6^2 = 36. \quad \textit{Ans.}$$

EXERCISES 57

Write each of the following equations in logarithmic form:

1. $2^5 = 32$ **2.** $3^4 = 81$ **3.** $49^{0.5} = 7$
4. $27^{4/3} = 81$ **5.** $10^4 = 10{,}000$ **6.** $10^{-5} = 0.00001$
7. $343^{1/3} = 7$ **8.** $64^{1/6} = 2$ **9.** $10^{0.30103} = 2$
10. $10^{1.47712} = 30$ **11.** $M = a^4$ **12.** $N = b^{-5}$

Write each of the following equations in exponential form:

13. $\log_7 49 = 2$ **14.** $\log_5 125 = 3$ **15.** $\log_3 \frac{1}{81} = -4$
16. $\log_6 \frac{1}{216} = -3$ **17.** $\log_{16} 8 = 0.75$
18. $\log_{32} 2 = 0.2$ **19.** $\log_{10} 200 = 2.30103$
20. $\log_{10} 4000 = 3.60206$ **21.** $\log_{10} 5 = 0.69897$

Find the value of each of the following logarithms:

22. $\log_3 27$ **23.** $\log_7 343$ **24.** $\log_{30} 900$
25. $\log_3 81$ **26.** $\log_2 64$ **27.** $\log_5 625$
28. $\log_4 \frac{1}{64}$ **29.** $\log_{10} 100{,}000$ **30.** $\log_{10} 0.000{,}001$

Find the value of the letter in each of the following equations:

31. $\log_3 x = 4$ **32.** $\log_5 w = 4$ **33.** $\log_{25} V = 1.5$
34. $\log_{19} u = 0.5$ **35.** $\log_{81} x = 0.75$ **36.** $\log_{10} N = -4$
37. $\log_{10} M = 5$ **38.** $\log_{625} R = 0.5$ **39.** $\log_{49} T = -1.5$
40. $\log_{128} t = -\frac{2}{7}$ **41.** $\log_{0.04} z = -2$ **42.** $\log_{0.01} V = -3$

Find the base of each of the following logarithms:

43. $\log_a 125 = 3$ **44.** $\log_b 64 = 6$ **45.** $\log_x 243 = 5$
46. $\log_y 64 = 3$ **47.** $\log_c 11 = 0.5$ **48.** $\log_w 27 = 0.6$
49. $\log_n 9 = \frac{2}{3}$ **50.** $\log_k 3 = \frac{1}{5}$ **51.** $\log_m 5 = -0.5$
52. $\log_d 10 = 0.2$ **53.** $\log_x 64 = -0.75$ **54.** $\log_b 0.1 = -0.5$

71. General Properties of Logarithms

From our work on exponents, it follows that, if b is any real, positive number and x is any rational number, the symbol b^x is fully defined.

For example,

$$5^2 = 25;$$

$$3^{-4} = \frac{1}{3^4} = \frac{1}{81};$$

and

$$(27)^{-2/3} = \frac{1}{(\sqrt[3]{27})^2} = \frac{1}{9}.$$

The meaning of such symbols as $5^{\sqrt{2}}$ and 3^{π} has not been defined. We shall assume here what is proved in much more advanced mathematics, namely:

(1) Irrational exponents can be so defined that they obey the same laws as rational exponents.

(2) By the use of enough digits from the decimal approximation to an irrational exponent, it is possible to make as close an approximation to the indicated power as is desired. Thus, $5^{\sqrt{2}}$ may be approximated as closely as necessary by the use of enough digits from the decimal approximation to $\sqrt{2}$. This means that $5^1, 5^{1.4}, 5^{1.41}, 5^{1.414}, 5^{1.4142}, \cdots$ is a sequence of constantly improving approximations to the number $5^{\sqrt{2}}$.

With these assumptions, it is clear that b^x will always stand for a definite number if b is positive and x is real. In fact, with the restriction to principal roots in case of fractional exponents, b^x is always a real positive number.

It can also be shown that if b is positive and not equal to 1 and N is positive, the equation $b^x = N$ has a single real solution for x. This result is a further assumption that we must make.

It is thus possible to use any real positive number except 1 as the base of a system of logarithms. However, since the only bases of actual importance are greater than 1, we shall restrict our discussion to this case. In the remainder of this chapter, the base b is to be understood to satisfy the condition $b > 1$.

The following properties and theorems are essential in the work with logarithms. (The student may observe that the restriction to bases greater than 1 is not always needed.)

(1) *The logarithm of 1 is zero.*

This property follows at once from the fact that $b^0 = 1$ $(b \neq 0)$.

(2) *The logarithm of the base is 1.*

Since $b^1 = b$, we have $\log_b b = 1$.

(3) *The logarithm of any number greater than 1 is positive; the logarithm of any number between 0 and 1 is negative.*

Illustrations: $\log_2 4 = 2$; $\log_3 27 = 3$; $\log_2 \frac{1}{4} = -2$; $\log_3 \frac{1}{27} = -3$.

(4) *The number zero has no logarithm.*

This is a restatement of the fact that $b^x = 0$ is satisfied by no value of x.

(5) *Negative numbers have no real logarithms.*

This statement is a consequence of the fact that any real power of a positive number is a positive number. The student should recall the restriction to principal roots in the case of fractional exponents.

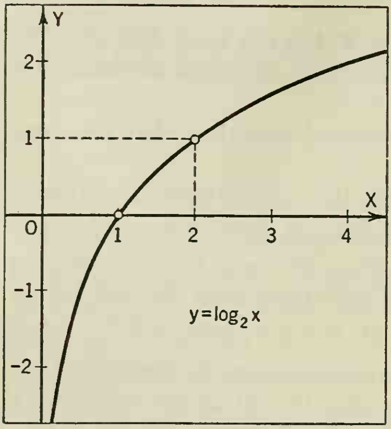


Fig. 40

NOTE: In advanced mathematics, logarithms are so defined that every number except zero has an unlimited set of logarithms to a given positive base. Each positive number, however, has exactly one real logarithm.

The five preceding properties are displayed in Figure 40, which is a graph of the equation $y = \log_2 x$.

The use of logarithms in computation depends upon the following three theorems which are restatements of the laws of exponents. It is to be understood that the same base is used throughout.

Theorem 1: *The logarithm of a product of two numbers is the sum of the logarithms of the factors.*

Thus, $$\log_b MN = \log_b M + \log_b N.$$

Proof: Let $x = \log_b M$ and $y = \log_b N$. Then, $b^x = M$ and $b^y = N$. Now,

$$MN = (b^x)(b^y) = b^{x+y}.$$

Hence, $$\log_b MN = x + y = \log_b M + \log_b N.$$

Illustration: $\log_2 4 = 2$ and $\log_2 8 = 3$. Therefore,

$$\log_2 32 = \log_2 [(4)(8)] = 2 + 3 = 5.$$

Theorem 2: *The logarithm of the quotient of two numbers is the logarithm of the dividend minus the logarithm of the divisor.*

Thus, $$\log_b \frac{M}{N} = \log_b M - \log_b N.$$

The proof, which is quite similar to that of Theorem 1, is left to the student.

Illustration: $\log_2 32 = 5$ and $\log_2 4 = 2$. Therefore,

$$\log_2 \tfrac{32}{4} = 5 - 2 = 3.$$

Check: $\log_2 \frac{32}{4} = \log_2 8 = 3$.

Theorem 3: *The logarithm of a power of a number is the product of the exponent and the logarithm of the number.*

$$\log_b M^p = p \log_b M.$$

The proof is again left to the student. HINT: $(b^x)^p = b^{px}$.

Illustrations: (*a*) $\log_2 4^3 = 3 \log_2 4 = 6$.

Check: $\log_2 64 = 6$.

$$(b)\ \log_3 \sqrt{81} = \log_3 (81)^{1/2} = \tfrac{1}{2} \log_3 81 = \tfrac{1}{2}(4) = 2.$$

Check: $\log_3 9 = 2$.

EXAMPLE 1. Show that

$$\log_b \sqrt{\frac{MN}{PQ}} = \frac{1}{2} (\log_b M + \log_b N - \log_b P - \log_b Q).$$

Solution:

$$\log_b \sqrt{\frac{MN}{PQ}} = \log_b \left(\frac{MN}{PQ}\right)^{1/2} = \frac{1}{2} \log_b \frac{MN}{PQ}. \qquad \text{(Theorem 3)}$$

$$\log_b \frac{MN}{PQ} = \log_b (MN) - \log_b (PQ). \qquad \text{(Theorem 2)}$$

$$\left.\begin{aligned} \log_b MN &= \log_b M + \log_b N. \\ \log_b PQ &= \log_b P + \log_b Q. \end{aligned}\right\} \qquad \text{(Theorem 1)}$$

Hence, $\log_b \sqrt{\dfrac{MN}{PQ}} = \dfrac{1}{2} (\log_b M + \log_b N - \log_b P - \log_b Q)$.

EXAMPLE 2. When the base 10 is used, the logarithm is abbreviated to *log* with no base indicated. Given the approximate values: $\log 2 = 0.30103$; $\log 3 = 0.47712$; $\log 5 = 0.69897$; find the corresponding values of (*a*) $\log 12$; (*b*) $\log \frac{25}{3}$; (*c*) $\log \sqrt{3}$.

Solution: (*a*) $\qquad 12 = (2^2)(3).$

Hence, $\qquad \log 12 = 2 \log 2 + \log 3.$

$$\begin{aligned} 2\log 2 &= 0.60206 \\ \text{Add} \qquad \log 3 &= \underline{0.47712} \\ \log 12 &= 1.07918. \quad \textit{Ans.} \end{aligned}$$

(*b*)
$$\frac{25}{3} = \frac{5^2}{3}.$$

Hence,
$$\log\frac{25}{3} = 2\log 5 - \log 3.$$

$$\begin{aligned} 2\log 5 &= 1.39794 \\ \text{Subtract} \qquad \log 3 &= \underline{0.47712} \\ \log\frac{25}{3} &= 0.92082. \quad \textit{Ans.} \end{aligned}$$

(*c*)
$$\sqrt{3} = 3^{1/2}.$$

Hence,
$$\log\sqrt{3} = \tfrac{1}{2}\log 3 = 0.23856. \quad \textit{Ans.}$$

EXERCISES 58

Given $\log 2 = 0.30103$, $\log 3 = 0.47712$, $\log 5 = 0.69897$, find the value of each of the logarithms listed. As in Example 2, above, the base is 10 when it is not indicated.

1. $\log 24$ **2.** $\log 72$ **3.** $\log 75$
4. $\log 360$ **5.** $\log \frac{36}{5}$ **6.** $\log \frac{27}{32}$
7. $\log 0.54$ **8.** $\log 0.025$ **9.** $\log 12^3$
10. $\log (1.35)^4$ **11.** $\log \sqrt{4.8}$ **12.** $\log \sqrt{150}$
13. $\log \sqrt[3]{7.2}$ **14.** $\log \sqrt[4]{1.25}$ **15.** $\log \sqrt[5]{0.16}$

Use the properties of logarithms to transform the left member into the right member in each of the following equations. Note that any positive base may be used except in Problems 26 and 27; these two equations are valid only if 10 is the base.

16. $\log \frac{11}{5} + \log \frac{14}{3} - \log \frac{22}{15} = \log 7$

17. $\log \frac{6}{7} - \log \frac{27}{4} + \log \frac{21}{16} = -\log 6$

18. $\log a^4 - \log \dfrac{6}{\sqrt[3]{a}} + \log 6\sqrt[3]{a^2} = 5 \log a$

19. $\log c^3 - \log \dfrac{2}{c^4} + \log \sqrt{c^3} + \log \dfrac{2}{\sqrt{c}} = 8 \log c$

20. $\frac{5}{2}\log h - 4\log k - \frac{1}{4}\log h + \frac{7}{4}\log k = \frac{9}{4}\log \dfrac{h}{k}$

21. $2\log c + \log d - \frac{5}{3}\log c - \frac{1}{6}\log d = \frac{1}{6}\log c^2d^5$

22. $\log \dfrac{(y-4)^{5/2}}{y^3(y^2-16)} = \frac{3}{2}\log (y-4) - \log (y+4) - 3\log y$

23. $\log \dfrac{x^3 + 8}{(x-2)^3 - 8} = \log (x + 2) - \log (x - 4)$

24. $\log \left(\dfrac{1}{z} - z\right)^4 = 4 \log (1 + z) + 4 \log (1 - z) - 4 \log z$

25. $\log \sqrt{\dfrac{x^2 - a^2}{(x + a)^2}} = \frac{1}{2} \log (x - a) - \frac{1}{2} \log (x + a)$

26. $\log \dfrac{(y + 3)^5 \cdot 10^{5y}}{10^{y^3}} = 5 \log (y + 3) + 5y - y^3$

27. $\log \dfrac{v^3 \cdot 10^{v^2}}{10^{2v}} = 3 \log v + v^2 - 2v$

28. $5 \log (w + 6) + \frac{1}{2} \log (w - 2) - \frac{9}{2} \log (w + 6) = \log \sqrt{w^2 + 4w - 12}$

29. $\log (5x - 2) + \frac{1}{6} \log (x + 7) - \frac{1}{3} \log (5x - 2) - \frac{2}{3} \log (x + 7)$

$$= \frac{1}{6} \log \frac{(5x - 2)^4}{(x + 7)^3}$$

30. $\log \dfrac{x}{c + \sqrt{c^2 + x^2}} = \log \dfrac{\sqrt{c^2 + x^2} - c}{x} = -\log \dfrac{c + \sqrt{c^2 + x^2}}{x}$

31. $\dfrac{1}{2} \log \dfrac{z + \sqrt{z^2 - d^2}}{z - \sqrt{z^2 - d^2}} + \log d = \log (z + \sqrt{z^2 - d^2})$

72. Logarithms to the Base 10

For purposes of computation, the most important set of logarithms is the one that uses the base 10; this system is called the **common** system of logarithms. We shall follow the customary procedure of omitting the base from the abbreviation *log* for common logarithms. Since 10 is the base, we have

$$\log 10 = 1.$$

Before attempting to understand how to use a table of common logarithms, it is necessary to consider the implications of the fact that any number written in ordinary decimal notation can be written as a product of a number between 1 and 10 and a power of 10. Thus, we may write

$$3967 = 3.967 \times 10^3;$$

$$287.2 = 2.872 \times 10^2;$$

$$3.34 = 3.34 \times 10^0;$$

$$0.98 = 9.8 \times 10^{-1};$$

$$0.00989 = 9.89 \times 10^{-3};$$

and so on.

As previously indicated in Section 69, we call this way of writing a number **standard notation.** In standard notation, *the decimal point is invariably placed immediately after the first digit on the left.* As a supplement to the discussion of Section 69, it is convenient to note that *the exponent of 10 has a numerical value equal to the number of places that the decimal point in the given number is displaced from standard position; the sign of the exponent is positive if the point is to the right, and negative if it is to the left of standard position.*

Illustrations: (*a*) In order to write 5289.6 in standard notation, we note that the decimal point is three places to the right of standard position; hence, $5289.6 = 5.2896 \times 10^3$.

(*b*) In the number 0.000523, the decimal point is four places to the left of standard position. Therefore, $0.000523 = 5.23 \times 10^{-4}$.

In standard notation, two numbers that have the same sequence of digits can differ only in the power of 10 that is used. Thus, the numbers

$$5376 = 5.376 \times 10^3;$$

$$53.76 = 5.376 \times 10^1;$$

$$0.5376 = 5.376 \times 10^{-1};$$

and

$$0.005376 = 5.376 \times 10^{-3}$$

differ only in the exponent applied to 10.

The logarithm of 5376 may now be written as

$$\log 5376 = \log(5.376 \times 10^3) = \log 5.376 + \log 10^3$$
$$= \log 5.376 + 3 \log 10 = \log 5.376 + 3.$$

Similarly,

$$\log 53.76 = \log 5.376 + 1;$$

$$\log 0.5376 = \log 5.376 - 1;$$

and

$$\log 0.005376 = \log 5.376 - 3.$$

We have

$$\log 1 = 0 \quad \text{and} \quad \log 10 = 1,$$

so that the logarithm of any number, say N, between 1 and 10 must be a number between 0 and 1, that is, $\log N$ is a decimal with no integral part. For instance, $\log 5.376 = 0.73046$, approximately. Hence, the common logarithm of any positive number can be written as the sum of two parts: (1) a decimal less than 1, and (2) the exponent applied to 10 when the number is written in the standard notation.

The decimal part of the common logarithm of a number is called

the **mantissa.** Our discussion shows that the *mantissa depends only on the particular sequence of digits in the number.* It is independent of the position of the decimal point.

The exponent of 10 which is the integral (whole-number) part of the logarithm is called the **characteristic.** *The characteristic depends only on the position of the decimal point in the number.* It is independent of the sequence of digits.

The rule for finding the exponent of 10 when writing the number in standard notation is exactly the rule for the determination of the characteristic of the logarithm. The student should study the following brief table to make certain he understands how the characteristics are obtained.

Number	Characteristic of Logarithm
34728.	4
92.345	1
3.681	0
0.278	−1
0.00532	−3

EXERCISES 59

Give the characteristic of the logarithm of each of the following numbers:

1. 4.1402 **2.** 520.82 **3.** 21076
4. 53.964 **5.** 0.37235 **6.** 0.08779
7. 1868.37 **8.** 0.00457 **9.** 0.12038
10. 784.11 **11.** 93.816 **12.** 4.57×10^4
13. 6.7432×10^{-5} **14.** 3.3648×10^{-4} **15.** 8.625×10^8

73. The −10 Notation for Negative Characteristics

In order to keep the mantissa of a logarithm positive, as is customary for ordinary computations, we do not combine the mantissa and characteristic when the latter is negative. Thus, if

$$\log 2 = 0.30103,$$

$$\log 0.002 = 0.30103 - 3.$$

This difference is ordinarily left in indicated form (rather than as −2.69897, which is the actual negative value). This is done in order to make easy reference to tables that give only positive mantissas.

N	0	1	2	3	4	5	6	7	8	9
250	39 794	811	829	846	863	881	898	915	933	950
251	967	985	*002	*019	*037	*054	*071	*088	*106	*123
252	40 140	157	175	192	209	226	243	261	278	295
253	312	329	346	364	381	398	415	432	449	466
254	483	500	518	535	552	569	586	603	620	637
255	654	671	688	705	722	739	756	773	790	807
256	824	841	858	875	892	909	926	943	960	976
257	993	*010	*027	*044	*061	*078	*095	*111	*128	*145
258	41 162	179	196	212	229	246	263	280	296	313
259	330	347	363	380	397	414	430	447	464	481
260	497	514	531	547	564	581	597	614	631	647
261	664	681	697	714	731	747	764	780	797	814
262	830	847	863	880	896	913	929	946	963	979
263	996	*012	*029	*045	*062	*078	*095	*111	*127	*144
264	42 160	177	193	210	226	243	259	275	292	308
265	325	341	357	374	390	406	423	439	455	472
266	488	504	521	537	553	570	586	602	619	635
267	651	667	684	700	716	732	749	765	781	797
268	813	830	846	862	878	894	911	927	943	959
269	975	991	*008	*024	*040	*056	*072	*088	*104	*120
270	43 136	152	169	185	201	217	233	249	265	281
271	297	313	329	345	361	377	393	409	425	441
272	457	473	489	505	521	537	553	569	584	600
273	616	632	648	664	680	696	712	727	743	759
274	775	791	807	823	838	854	870	886	902	917
275	933	949	965	981	996	*012	*028	*044	*059	*075
276	44 091	107	122	138	154	170	185	201	217	232
277	248	264	279	295	311	326	342	358	373	389
278	404	420	436	451	467	483	498	514	529	545
279	560	576	592	607	623	638	654	669	685	700
280	716	731	747	762	778	793	809	824	840	855
281	871	886	902	917	932	948	963	979	994	*010
282	45 025	040	056	071	086	102	117	133	148	163
283	179	194	209	225	240	255	271	286	301	317
284	332	347	362	378	393	408	423	439	454	469
285	484	500	515	530	545	561	576	591	606	621
286	637	652	667	682	697	712	728	743	758	773
287	788	803	818	834	849	864	879	894	909	924
288	939	954	969	984	*000	*015	*030	*045	*060	*075
289	46 090	105	120	135	150	165	180	195	210	225
290	240	255	270	285	300	315	330	345	359	374
291	389	404	419	434	449	464	479	494	509	523
292	538	553	568	583	598	613	627	642	657	672
293	687	702	716	731	746	761	776	790	805	820
294	835	850	864	879	894	909	923	938	953	967
295	982	997	*012	*026	*041	*056	*070	*085	*100	*114
296	47 129	144	159	173	188	202	217	232	246	261
297	276	290	305	319	334	349	363	378	392	407
298	422	436	451	465	480	494	509	524	538	553
299	567	582	596	611	625	640	654	669	683	698
300	712	727	741	756	770	784	799	813	828	842
N	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	18
1	1.8
2	3.6
3	5.4
4	7.2
5	9.0
6	10.8
7	12.6
8	14.4
9	16.2

	17
1	1.7
2	3.4
3	5.1
4	6.8
5	8.5
6	10.2
7	11.9
8	13.6
9	15.3

	16
1	1.6
2	3.2
3	4.8
4	6.4
5	8.0
6	9.6
7	11.2
8	12.8
9	14.4

	15
1	1.5
2	3.0
3	4.5
4	6.0
5	7.5
6	9.0
7	10.5
8	12.0
9	13.5

	14
1	1.4
2	2.8
3	4.2
4	5.6
5	7.0
6	8.4
7	9.8
8	11.2
9	12.6

Prop. Parts

It has become a fairly uniform practice among computers to add and subtract a number from the logarithm so that it is written as *a positive number minus 10 or an integral multiple of 10.* Thus, the previous example would be written

$$\begin{aligned}\log 0.002 &= 0.30103 - 3 \\ &= 7.30103 - 10.\end{aligned}$$

Observe that 7 has been added and subtracted so that only the form and not the value of the logarithm has been changed. It is apparent that any such change is legitimate. In fact, if $\log \sqrt[3]{0.002}$ is to be found, we may write

$$\begin{aligned}\log \sqrt[3]{0.002} &= \tfrac{1}{3} \log 0.002 = \tfrac{1}{3}(0.30103 - 3) = \tfrac{1}{3}(27.30103 - 30) \\ &= 9.10034 - 10.\end{aligned}$$

The number 27 has been added and subtracted in order that the division may yield the -10 form. In this last result, the characteristic is $9 - 10$. The student should examine the following brief table to observe how characteristics may be written in the -10 form.

Number	Characteristic of Logarithm
0.25	9 − 10
0.0378	8 − 10
0.000069	5 − 10

EXERCISES 60

Give the characteristic of the logarithm of each of the following numbers. Use the -10 notation.

1. 0.83927 **2.** 0.00284 **3.** 0.06926
4. 0.00004 **5.** 0.07525 **6.** 0.31713
7. 0.05079 **8.** 1.8467×10^{-2} **9.** 3.9052×10^{-1}
10. 6.1109×10^{-3} **11.** 2.639×10^{-4} **12.** 5.0046×10^{-5}

74. The Use of Tables

The opposite page is reprinted from Table II (see Appendix), which gives five-place mantissas for the numbers from 1.000 to 9.999 at intervals of 0.001. The principles involved in the use of these tables are the same as for four-, six-, or ten-place logarithm tables, as well as for many other tables that the computer will use.

Decimal points are customarily omitted in such tables; they are to be supplied by the user. The first three digits of the number are given in the column at the left under the caption N. The fourth digit of the number occurs at the top of the table in the same line as N. The first entry in the opposite table gives log 2.500 = 0.39794. The next entry in the same line is log 2.501 = 0.39811. In each case, the mantissa obtained directly from the table is the entire logarithm, since the characteristic is zero. Notice that the first two digits of the logarithm are printed only in the first column of mantissas, and then only when a change occurs. For example, from 2.510 to 2.520 there is a change from 39 to 40 for the first two digits in the mantissa.

In order to read log 2.846, follow down the column under N to the number 284. Go across the line to the column under 6. Then read

$$\log 2.846 = 0.45423.$$

In order to read log 2.957, the same procedure is used. Notice this time that the last three digits of the mantissa are starred; the star means that the first two digits are to be read from the *following* line of the table. Thus,

$$\log 2.957 = 0.47085.$$

The number that corresponds to a given logarithm is often referred to as the **antilogarithm**. For example, the last displayed statement may be written

$$\text{antilog } 0.47085 = 2.957.$$

The instructions for reading antilogarithms are exactly the reverse of those already given for logarithms. Thus, antilog 0.42749 is found by following down the first column of mantissas until the digits 42 occur. An examination of this section of the table shows 42749 in the line with $N = 267$ and in the column under 6. Hence,

$$\text{antilog } 0.42749 = 2.676.$$

In all the illustrations above, the characteristics have been zeros. When this is not the case, the location of the decimal point in the number and the determination of the characteristic of its logarithm must be handled according to the discussion in the preceding sections.

EXAMPLE 1. Find (*a*) log 277.8; (*b*) log 0.02778.

Solution: Since these two numbers have the same sequence of digits 2778, their logarithms will have the same mantissas. From the table,

we find log 2.778 = 0.44373. Hence,

(*a*) $\log 277.8 = 2.44373$. *Ans.*

(*b*) $\log 0.02778 = 8.44373 - 10$. *Ans.*

EXAMPLE 2. Find (*a*) antilog 3.43072; (*b*) antilog 6.43072 − 10.

Solution: Both the logarithms have the mantissa 0.43072, which from the table is found to have an antilogarithm of 2.696. Hence,

(*a*) $\text{antilog } 3.43072 = 2696$. *Ans.*

(*b*) $\text{antilog } 6.43072 - 10 = 0.0002696$. *Ans.*

EXERCISES 61

From Table II read the logarithm of each of the following numbers:

1. 74.48
2. 952.0
3. 1837
4. 3.046
5. 0.04371
6. 50.18
7. 0.1283
8. 3.632
9. 164,200
10. 257.5
11. 0.008606
12. 0.01004
13. 0.06737
14. 9.444×10^{-4}
15. 8.319×10^{5}

From Table II read the value of N in each of the following:

16. $\log N = 1.26764$
17. $\log N = 0.44091$
18. $\log N = 2.57967$
19. $\log N = 3.65040$
20. $\log N = 0.76716$
21. $\log N = 9.83046 - 10$
22. $\log N = 8.03463 - 10$
23. $\log N = 7.31154 - 10$
24. $\log N = 4.68178$
25. $\log N = 8.98014 - 10$
26. $\log N = 1.16673$
27. $\log N = 2.56205$
28. $\log N = 0.85022$
29. $\log N = 6.94002 - 10$
30. $\log N = 9.32777 - 10$
31. $\log N = 1.00130$

75. Interpolation

A five-place table of logarithms can be used to find mantissas of five-digit numbers with sufficient accuracy for ordinary purposes. The process of placing numbers between successive entries of the table is known as **interpolation.**

Suppose it is required to find log 2.7243. We know that this logarithm must lie between log 2.7240 and log 2.7250. Since a very short section of the logarithmic curve is involved here, we obtain a sufficiently

good approximation by replacing the curve by a straight line. This is very much magnified in Figure 41.

Let h be the number which must be added to log 2.7240 to give log 2.7243. Using the similar triangles PQS and PRT, in the figure, we have

$$\frac{h}{d} = \frac{2.7243 - 2.7240}{2.7250 - 2.7240} = \frac{3}{10},$$

that is, $h = \frac{3}{10}d.$

The number d is called a **tabular difference**; it is the difference between two successive entries in the table. The use of the similar triangles is equivalent to the assumption that the differences in the numbers are proportional to the differences in their logarithms.

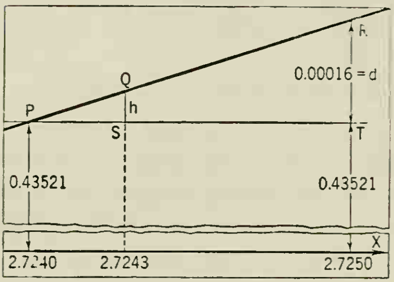


Fig. 41

Omitting the decimal points, we have

$$d = 43537 - 43521 = 16$$

and $h = \frac{3}{10}d = 4.8.$

Since the table gives no information about the sixth digit of the mantissa, it would be pointless to extend any logarithm beyond five places. We therefore round off 4.8 to 5 and add this amount to 43521 to obtain

$$\log 2.7243 = 0.43526.$$

In the right margin of the table, the student will see tabulations of proportional parts for tabular differences occurring on the same pages. For the difference 16, the numbers under 16 are, respectively, $\frac{1}{10}$, $\frac{2}{10}$, $\frac{3}{10}$, $\cdots$, $\frac{9}{10}$ of 16 as indicated by the boldface numbers at the left.

Example 1. Find log 2.9357.

Solution: From the table, omitting decimal points, we have

$$10\left[\begin{array}{cc} & \text{Number} \quad\quad \text{Mantissa} \\ 7\left[\begin{array}{l}29350 \\ 29357\end{array}\right. & \left.\begin{array}{r}46761 \\ 467??\end{array}\right] x \\ \quad 29360 & 46776\end{array}\right] 15$$

From the proportional parts table under 15, we read $\frac{7}{10}(15) = 10.5.$

This is added to 46761 to give 46771.5, which is rounded off to give

$$\log 2.9357 = 0.46772. \quad \textit{Ans.}$$

In this example, the rounding off is done so that the *final result* ends in an even digit. This procedure is in accord with the rules on page 110.

Antilogarithms of mantissas that do not occur in the table can be found by reversing the interpolation process described above. For example, to find antilog 0.44800, we first locate two consecutive mantissas, one on each side of 44800. From the table, we have

$$\begin{array}{cccc}
 & \text{Mantissa} & \text{Number} & \\
16 \left[\begin{array}{l} 7\left[\begin{array}{l}44793 \\ 44800\end{array}\right. \\ \quad 44809 \end{array}\right. & & \left.\begin{array}{r} \left.\begin{array}{r}28050 \\ 2805?\end{array}\right] x \\ 28060 \quad \end{array}\right] 10 &
\end{array}$$

By using the same idea of proportion as before, we find

$$\frac{x}{10} = \frac{7}{16} \quad \text{or} \quad x = 4.4-.$$

Therefore, $\qquad$ antilog 0.44800 = 2.8054.

All antilogarithms should be rounded off to not more than the number of digits given in the mantissa.

The result of the preceding example can be checked by inspection of the proportional-parts table for $d = 16$. Under 16, the number nearest the difference 7 is 6.4, which is opposite 4. Hence, 4 is the fifth digit of the antilogarithm, as was found above.

Example 2. Find antilog 0.40251.

Solution: From the table, we find

$$\begin{array}{ccc}
 & \text{Mantissa} & \text{Number} \\
18 \left[\begin{array}{l} 8\left[\begin{array}{l}40243 \\ 40251\end{array}\right. \\ \quad 40261 \end{array}\right. & & \begin{array}{r} \left.\begin{array}{r}25260 \\ 2527?\end{array}\right] x \\ 25270 \quad \end{array}
\end{array}$$

In the proportional-parts table under 18, the number nearest 8 is 7.2, which is opposite 4. Hence, $x = 4$, and

$$\text{antilog } 0.40251 = 2.5264. \quad \textit{Ans.}$$

As a general rule, in writing logarithms, the student should write

the characteristic first and then employ the table to find the mantissa. Interpolation is to be used when necessary and should be done mentally.

EXERCISES 62

Find the logarithm of each of the following numbers:

1. 471.55	**2.** 72.658	**3.** 8042.3
4. 2.4236	**5.** 0.12574	**6.** 57,579
7. 0.036361	**8.** 2.0067×10^{-3}	**9.** 988.36
10. 0.77294	**11.** 10.942	**12.** 2.6398×10^{-2}
13. 6.1852×10^{-5}	**14.** 3.5004×10^{-4}	**15.** 1.0297×10^{6}

Find the value of N to 5 significant digits in each of the following:

16. $\log N = 0.35240$	**17.** $\log N = 1.65517$
18. $\log N = 3.49283$	**19.** $\log N = 9.74431 - 10$
20. $\log N = 8.88016 - 10$	**21.** $\log N = 2.13395$
22. $\log N = 0.40117$	**23.** $\log N = 7.84533 - 10$
24. $\log N = 1.54890$	**25.** $\log N = 6.95024 - 10$
26. $\log N = 4.23535$	**27.** $\log N = 8.58776 - 10$
28. $\log N = 9.00175 - 10$	**29.** $\log N = 3.52468$
30. $\log N = 2.87562$	**31.** $\log N = 0.22027$

76. Computations with Logarithms

At this point, the student should again make sure that he understands the theorems on combination of logarithms. By means of these theorems, multiplication is replaced by addition, division by subtraction, and raising to a power by simple multiplication.

Before any computation is done in a problem, it is important to analyze the problem and to construct a well-thought-out form for the calculation. The characteristic of the logarithm of each given number should be written in the proper location in the form as is illustrated in the following examples.

EXAMPLE 1. Find the value of $(0.6532)(92.94)(214.1)$.

Solution: Let $x = (0.6532)(92.94)(214.1)$.

Then, $\log x = \log a + \log b + \log c,$

where a, b, and c denote the three factors, respectively.

A convenient form for performing this computation is the first one

below. This is filled in with the data that are available before using the tables.

	a	0.6532	log	9. − 10
	b	92.94	log	1.
	c	214.1	log	2.
Ans.	x		log	

The same form, completely filled in, appears next.

	a	0.6532	log	9.81505 − 10
	b	92.94	log	1.96820
	c	214.1	log	2.33062
Ans.	x	12998	log	4.11387

Example 2. Find the value of $\dfrac{(0.30125)(6.2457)}{(93.468)(0.04837)}$.

Solution: Let $x = \dfrac{(0.30125)(6.2457)}{(93.468)(0.04837)} = \dfrac{ab}{cd} = \dfrac{N}{D}$.

Then,
$$\log x = \log N - \log D,$$
$$\log N = \log a + \log b,$$
and
$$\log D = \log c + \log d.$$

Again, a suitable form, not completely filled in, is given.

a	0.30125	log	9. −10	c	93.468	log	1.
b	6.2457	log	0.	d	0.04837	log	8. −10
N	.....	log		D	.....	log	
		(−)					
D	.....	log					
Ans. x		log					

The dotted lines are put in spaces where the number is not required. The form represents exactly the analysis of the computation which was made above. Notice the symbol (−) which occurs between log N and log D in the first column of logarithms. This is to remind us that a subtraction is to be performed; addition is to be understood where (−) is not used. It is always most efficient to use the tables to find the man-

tissas for all the given numbers before any other computations are done.
The computation appears in completed form below.

a	0.30125	log	9.47892 − 10	c	93.468	log	1.97066
b	6.2457	log	0.79558	d	0.04837	log	8.68458 − 10
N	.....	log	0.27450	D	.....	log	0.65524
		(−)					
D	.....	log	0.65524				
Ans. x	0.41616	log	9.61926 − 10				

In the subtraction of log D from log N, we have regarded log N as 10.27450 − 10; this device keeps the decimal part of log x positive.

EXAMPLE 3. Find the value of $(0.7234)^5$.

Solution: Let $x = (0.7234)^5 = a^5$.

Then, $\log x = 5 \log a$.

The computation is as follows:

	a	0.7234	log	9.85938 − 10
				× 5
Ans.	x	0.19811	log	9.29690 − 10

When log a is multiplied by 5 to obtain log x, the characteristic comes out 49 − 50 which is mentally reduced to 9 − 10.

EXAMPLE 4. Calculate the value of $\sqrt{0.11023}$.

Solution: Let $\sqrt{0.11023} = \sqrt{a}$.

Then, $\log \sqrt{a} = \frac{1}{2} \log a$.

	a	0.11023	log	19.04230 − 20
				× $\frac{1}{2}$
Ans.	$\sqrt{a}$	0.33201	log	9.52115 − 10

The characteristic of log a was written 19 − 20 in preparation for the division by 2. This is to make the result appear in −10 notation.

EXAMPLE 5. Find the value of $\dfrac{\sqrt[5]{-992.7}}{\sqrt[3]{0.2358}}$.

Solution: The required result is negative by inspection. We temporarily disregard signs and let $x = \dfrac{\sqrt[5]{a}}{\sqrt[3]{b}}\cdot$

Then, $\log x = \frac{1}{5}\log a - \frac{1}{3}\log b.$

a	992.7	log	2.99682	$\frac{1}{5}$ log	0.59936
				(−)	
b	0.2358	log	29.37254 − 30	$\frac{1}{3}$ log	9.79085 − 10
x	6.4344	...	....	log	0.80851

Hence, the required result is −6.4344. *Ans.*

Example 6. Find the approximate value of $(1.03)^{20}$.

Solution:

	a	1.03	log	0.01284
				×20
Ans.	x	1.806	log	0.2568

Comments: (1) In the last example, the answer is given to only four significant digits even though 1.03 may be considered as exact. The multiplication of log 1.03 by as large a number as 20 leaves the fifth decimal place in log x unknown. This statement is in accord with our discussion of computation with approximate numbers, since log 1.03 $\doteq$ 0.01284 is an approximation that has been given correct to five decimal places, but to only four significant digits.

(2) As a general rule, five-place mantissas allow not more than five significant digits in the result of a logarithmic computation, even though the data are exact. However, the computer should follow the rules thathave been given for handling approximate numbers.

(3) If a number with more than five significant digits occurs in the data, many computers round off the number to five significant digits before interpolating in the table. The answers for all logarithmic computations in this book were obtained by following this procedure.

(4) Although negative numbers have no real logarithms, we can read by inspection the sign of the result of a computation as was done in Example 5. The computation may then be accomplished as though all numbers were positive, and the correct sign may be prefixed to the answer.

EXERCISES 63

Use logarithms to perform the following computations. Assume that the given numbers are exact and obtain each answer as accurately as you can.

1. $(6.544)(10.98)$ **2.** $(97.53)(8115)$

3. $(-7.286)(184.37)$ **4.** $(0.0034851)(39{,}520)$

5. $(-23{,}594)(0.00048532)$ **6.** $(-0.0026312)(-45.78)$

7. $67.42 \div 57.99$ **8.** $32.33 \div 435.67$

9. $-0.0043227 \div 0.94538$ **10.** $-8.1525 \div 0.0025461$

11. $(402.14)(0.0071863)(1.3647)$

12. $(0.11179)(0.066312)(8.4513)$

13. $(82{,}074)(-0.074313)(-0.0016139)$

14. $\dfrac{(60.13)(422.9)}{(318.5)(2.838)}$ **15.** $\dfrac{(-3.674)(-0.4863)}{(0.0005198)(121.1)}$

16. $\dfrac{(0.09124)(-7{,}560.3)}{(0.16342)(37.45)}$ **17.** $\dfrac{(52{,}187)(0.0034716)}{(14{,}693)(81{,}432)}$

18. $(2.357)^2$ **19.** $(3.418)^3$ **20.** $(10.057)^4$

21. $(0.04817)^2$ **22.** $(0.79132)^3$ **23.** $(7.5103)^5$

24. $\sqrt{51.816}$ **25.** $\sqrt{0.13221}$ **26.** $\sqrt[3]{-119.03}$

27. $\sqrt[3]{0.0084137}$ **28.** $\sqrt[5]{0.032469}$ **29.** $(64.45)^{3/5}$

30. $(0.012745)^{1/6}$ **31.** $(0.4231)^{3/10}$ **32.** $(0.21516)^{3/7}$

33. $(25.109)^{0.2}$ **34.** $(25.109)^{-0.2}$ **35.** $(0.018372)^{0.6}$

36. $\sqrt[3]{(100.05)(469.13)}$ **37.** $\sqrt[4]{(683.34)(0.2456)}$

38. $\sqrt[3]{2.57}\,\sqrt{8.92}$ **39.** $\sqrt[3]{-0.0506}\,\sqrt[4]{0.1234}$

40. $\dfrac{\sqrt[4]{91.85}}{\sqrt{35{,}976}}$ **41.** $\dfrac{\sqrt{0.05319}}{\sqrt[5]{-0.2032}}$ **42.** $\dfrac{\sqrt[3]{842.9}}{\sqrt[5]{-4373}}$

43. $\sqrt{\dfrac{(32.53)(2.068)}{68.53}}$ **44.** $\sqrt[3]{\dfrac{(122.74)(0.05612)}{\pi^2}}$

45. $(4.92)^{3.21}$ **46.** $(61.38)^{0.146}$ **47.** $(0.0724)^{-3.04}$

48. $(0.347)^{-0.562}$ **49.** $(0.347)^{0.562}$ **50.** $(0.0724)^{3.04}$

NOTE: Additional problems involving logarithmic computation may be taken from the exercises at the end of the next section.

77. Cologarithms

Computations involving division can be further simplified by making use of the fact that division by a number is equivalent to multiplication by its reciprocal. Thus, if

$$x = \frac{MN}{PQ},$$

x may be thought of as the product

$$(M)(N)\left(\frac{1}{P}\right)\left(\frac{1}{Q}\right).$$

Hence, $$\log x = \log M + \log N + \log \frac{1}{P} + \log \frac{1}{Q}.$$

The logarithm of the reciprocal of a number is called the **cologarithm** of the number, that is,

$$\mathbf{\log \frac{1}{P} = colog\ P.}$$

Furthermore, the cologarithm of a number may be read from the table almost as easily as the logarithm itself, for

$$\text{colog } P = \log \frac{1}{P} = \log 1 - \log P.$$

Since $\log 1 = 0 = 10 - 10$, we have

$$\text{colog } P = 10 - \log P - 10.$$

(The device of writing $10 - 10$ in place of 0 is used to keep the decimal part of the cologarithm positive.)

In order to find colog P, we have only to subtract log P from $10 - 10$. This can easily be accomplished mentally as is explained in the following examples.

Example 1. Find colog 27.32.

Solution: In the table we find log 27.32 = 1.43648, which is to be subtracted from $10 - 10$. The result is

$$\text{colog } 27.32 = 8.56352 - 10. \quad \textit{Ans.}$$

The student's mental subtraction is aided by visualizing the first 10 as 9.9999(10) written above log 27.32 in the table; thus,

$$\begin{array}{l} 9.9999(10) - 10 \\ \underline{1.4364\ \ 8\qquad\quad} \\ 8.5635\ \ 2\ - 10. \end{array}$$

This corresponds to the arithmetic process of "borrowing" and enables the computer to write the cologarithm from left to right by subtracting each digit, except the last one in the logarithm, from 9. The last *nonzero* digit is to be subtracted from 10.

Example 2. Find colog 0.2904.

Solution: Using the tables, we have

$$\left[\begin{array}{rl} & 9.99(10)00 - 10 \\ \log 0.2904 = & \underline{9.46\ \ 3\ 00 - 10} \end{array}\right]$$

$$\text{colog } 0.2904 = 0.53\ \ 7\ 00. \qquad \textit{Ans.}$$

The bracketed lines should not be written down. The student should practice until he can write cologarithms accurately and with facility directly from the table.

Example 3. Using cologarithms, compute the value required in Example 2, of the preceding section.

Solution:

	a	0.30125	log	9.47892 − 10
	b	6.2457	log	0.79558
	c	93.468	colog	8.02934 − 10
	d	0.04837	colog	1.31542
Ans.	x	0.41616	log	9.61926 − 10

The student should compare this computation with the previous calculation for the same problem. Notice the advantages gained by using cologarithms:

(1) The numbers N and D are not needed.

(2) There is no subtraction of logarithms.

(3) The entire computation is handled in a simpler schedule of operations.

Example 4. Compute the value of $\dfrac{(2.3725)\ (34.69)^{2/3}}{(46.7)\ (0.5926)^{1/4}}$.

Solution:

	a	2.3725	...	.....	log	0.37520
	b	34.69	log	1.54020	$\frac{2}{3}$ log	1.02680
	c	46.7	...	.....	colog	8.33068 − 10
	d	0.5926	colog	0.22724	$\frac{1}{4}$ colog	0.05681
Ans.	x	0.61587	...	.....	log	9.78949 − 10

Note: There is no advantage gained by the use of cologarithms unless more than two numbers are involved in the computation. For

example, it is hardly worth while to use cologarithms to divide one number by another.

EXERCISES 64

Perform the computations in Examples 1 to 16 with the use of cologarithms.

1. $\dfrac{(493.71)(18.804)}{(6.1739)(51.265)}$

2. $\dfrac{(3.6508)(8095.7)}{(0.72854)(958.09)}$

3. $\dfrac{96{,}433}{(0.071479)(61{,}331)(0.82613)}$

4. $\dfrac{58{,}417}{(29.528)(0.80136)(37{,}048)}$

5. $\dfrac{(17{,}407)(406.15)(0.83824)}{(74{,}293)(0.012457)(3231.1)}$

6. $\dfrac{(64.296)(8.7532)(0.32923)}{(0.00716)(5.8024)(984.17)}$

7. $\sqrt{\dfrac{(0.2334)(471.62)}{(1.6049)(9.3826)}}$

8. $\sqrt[3]{\dfrac{(0.016158)(8.3547)}{(3.0014)(\pi)}}$

9. $\sqrt[4]{\dfrac{(1728.4)(38.405)}{(0.051143)(952.98)}}$

10. $\sqrt[5]{\dfrac{(466.25)(0.28491)}{(4585.2)(147.02)}}$

11. $\dfrac{(43.1)^2}{(1.728)^4(9.516)^3}$

12. $\dfrac{(16.47)^4}{(2.843)^3(0.3415)^2}$

13. $\left[\dfrac{576.4}{(4.8133)(10.28)}\right]^{3/4}$

14. $\left[\dfrac{(34.78)(6.173)}{(580.9)(44{,}170)}\right]^{2/5}$

15. $\dfrac{\sqrt{38{,}350}}{\sqrt[4]{0.14832}\,\sqrt[3]{73.618}}$

16. $\dfrac{\sqrt[3]{13{,}467}}{(2.6294)^4(496.25)^{2/3}}$

In each of the following problems, the data are to be considered as accurate to the stated number of digits. Perform the computations with five-place logarithms, then round off each answer to the number of significant digits allowed by the data.

17. A steel rod is 0.6250 in. in diameter and 60.00 in. long. If steel weighs 489 lb per cu ft, calculate the weight of the rod.

18. How many steel ball bearings 0.3125 in. in diameter will it take to weigh 1 lb? (A cu ft of steel weighs 489 lb.)

19. A spherical shell whose outside diameter is 6.00 in. is made of copper sheeting 0.0625 in. thick. Calculate the weight of the shell if copper weighs 556 lb per cu ft.

20. The area of a triangle may be calculated by using the formula

$$A = \sqrt{s(s - a)(s - b)(s - c)},$$

where s is one half of the perimeter and a, b, and c are the sides of the triangle. Calculate the area when $a = 733.6$ ft, $b = 871.4$ ft, and $c = 328.8$ ft.

21. A formula for the horsepower H developed by a steam engine is $H = \dfrac{PLAN}{33{,}000}$, where the denominator is exact and the numerator consists of

measured quantities. Calculate H if $P = 150$, $L = 3.00$, $A = 300$, and $N = 146$.

22. The period (time for one complete oscillation) of a simple pendulum is given by the approximate formula $t = 2\pi\sqrt{\dfrac{l}{32.16}}$, where l is the length of the pendulum in feet and t is the time in seconds. Find the period of a pendulum 8.2083 ft long.

23. In thermodynamics, the volume V and the pressure P of a given mass of air are frequently assumed to obey the equation $PV^{1.405} = C$, where C is a constant. If 27.48 cu ft of air at a pressure of 14.72 lb per sq in. is compressed to a pressure of 32.3 lb per sq in., what is the new volume if the above equation holds?

78. Natural Logarithms and Change of Base

Besides the system of common logarithms, there is another system, which is of practical importance. This is the system of *natural* logarithms whose base is the irrational number $2.71828\cdots$, denoted by the letter e. It is shown in calculus why e is used in this manner. It will have to suffice here to state that natural logarithms occur in many physical and engineering problems, as well as in many branches of theoretical mathematics. In place of $\log_e$, we shall use the abbreviation **ln** for natural logarithm.

The relation between logarithms to different bases can be obtained directly from the definition of a logarithm. It follows first from the properties of exponents that any positive number a may be considered as a power of a second positive number b $(b \neq 1)$, that is,

$$a = b^k. \tag{1}$$

Next, suppose that a and b are used as bases for two systems of logarithms. Then, for any positive number N, let

$$\log_b N = x \quad \text{or} \quad b^x = N; \tag{2}$$

and

$$\log_a N = y \quad \text{or} \quad a^y = N. \tag{3}$$

By equating the expressions for N in (2) and (3), we have

$$b^x = a^y.$$

If we now replace a by b^k from Equation (1), we find

$$b^x = (b^k)^y = b^{ky}.$$

Therefore,

$$x = ky. \tag{4}$$

The last equation is an important result which may be stated in the words: *The logarithms to a specified base are proportional to the corresponding logarithms to any other base.*

In order to find the relation between logarithms to specific bases, we need only evaluate k in Equation (4). This evaluation can be accomplished by using for x and y any pair of corresponding logarithms, one from each system.

For the case where the two bases are 10 and e, we have from (4),

$$\log N = k \ln N.$$

Therefore, if we choose $N = e$,

$$\log e = k \ln e,$$

or, since $\ln e = 1$,

$$k = \log e.$$

From the tables we find

$$\log e = \log 2.71828 = 0.43429.$$

Hence,

$$\log N = \mathbf{0.43429} \ln N. \tag{5}$$

If Equation (5) is solved for $\ln N$, we find

$$\ln N = \mathbf{2.3026} \log N. \tag{6}$$

Equations (5) and (6) are the equations for converting natural logarithms to common and common logarithms to natural, respectively. The numerical coefficients are both correct to five significant digits.

An inspection of Table III (see Appendix), which is a short table of natural logarithms, shows that no system of characteristics and mantissas is involved as there is in our analysis of common logarithms. This is made clear from the fact that $\ln 10 = 2.3026$, which may be verified by letting $N = 10$ in Equation (6). This result shows that both the integral and the decimal parts of natural logarithms are involved in the placement of the decimal point of the antilogarithm. Because of this somewhat more complicated situation, natural logarithms are not used for computations of the type considered in the preceding sections of this chapter; they are of value chiefly in various theoretical considerations.

Example 1. Find ln 25 by using five-place common logarithms.

Solution: From Equation (6), we have

$$\ln 25 = 2.3026 \log 25.$$

Table II gives log 25 = 1.39794, so that

$$\ln 25 = (2.3026)\,(1.39794).$$

a	2.3026	log	0.36222
b	1.3979	log	0.14548
x	3.2188	log	0.50770

Hence, $\ln 25 = 3.2188.$ *Ans.*

EXAMPLE 2. Find ln 25 by using Table III.

Solution: Since

$$\ln 25 = \ln (2.5)\,(10) = \ln 2.5 + \ln 10,$$

we have from Table III

$$\begin{aligned} \ln 2.5 &= 0.9163 \\ \ln\ 10 &= 2.3026 \\ \hline \end{aligned}$$

Hence, $\ln\ 25 = 3.2189.$ *Ans.*

This answer is slightly more accurate than the preceding one. The discrepancy is accounted for by the fact that the use of five-place tables does not guarantee five-figure accuracy.

Example 2 illustrates the practical method for finding natural logarithms of numbers lying outside the range of the table. In general, if x does not lie in the correct range, we write

$$x = (N)(10^m),$$

and

$$\begin{aligned} \ln x &= \ln N + m \ln 10 \\ &= \ln N + m(2.3026), \end{aligned}$$

where m is a positive or negative integer so chosen that N is a number that lies in the range of the table. The value of ln x can then be obtained by adding m times 2.3026 to the tabular value of ln N.

EXAMPLE 3. Use both methods thus far described to find ln 0.012.

Solution—First Method: Using common logarithms we have

$$\begin{aligned} \ln 0.012 &= 2.3026 \log 0.012 \\ &= (2.3026)(8.07918 - 10) \\ &= (2.3026)(-1.92082). \end{aligned}$$

a	2.3026	log	0.36222
b	1.9208	log	0.28348
x	4.4228	log	0.64570

Hence, $\ln 0.012 = -4.4228$. *Ans.*

Second Method: Using Table III, we write

$$\ln 0.012 = \ln (1.2)(10^{-2})$$
$$= \ln 1.2 - 2 \ln 10;$$
$$-2 \ln 10 = -4.6052;$$
$$\ln 1.2 = 0.1823.$$

Therefore, $\ln 0.012 = -4.4229$. *Ans.*

EXAMPLE 4. Find the value of P if $\ln P = 3.9375$. Use two methods.

Solution—First Method: From Equation (5), we have

$$\log P = 0.43429 \ln P$$
$$= (0.43429)(3.9375).$$

	a	0.43429	log	9.63778 − 10		
	b	3.9375	log	0.59522		
Ans.	P	51.29	log log	0.23300	log	1.7100

In this computation, the addition of log a and log b gave log (log P). The antilogarithm of this result is log P. In using logarithms in this manner, the student should bear in mind that logarithms are numbers with properties like any other numbers.

Second Method: Examination of Table III shows that 3.9375 is larger than any logarithm tabulated there. This fact means that P is greater than 10. Hence, we must subtract ln 10 from ln P enough times to give a value in the tabulated range. Thus,

$$\ln P - \ln 10 = 3.9375 - 2.3026$$
$$= 1.6349.$$

Also, $$\ln P - \ln 10 = \ln \frac{P}{10},$$

so that $$\ln \frac{P}{10} = 1.6349.$$

By interpolating in the table, we find

$$\frac{P}{10} = 5.129.$$

Hence, $$P = 51.29. \quad \textit{Ans.}$$

EXERCISES 65

Use the two methods of this section to find the approximate value of each of the following natural logarithms:

1. $\ln 28.41$ **2.** $\ln 2.841$ **3.** $\ln 15.36$
4. $\ln 0.05234$ **5.** $\ln 0.1273$ **6.** $\ln 0.0004$
7. $\ln \pi^2$ **8.** $\ln \frac{1}{7}$ **9.** $\ln \sqrt[3]{15.18}$

Find the value of x in each of the following equations in two ways:

10. $\ln x = 0.8850$ **11.** $\ln x = 1.5426$
12. $\ln x = 2.0763$ **13.** $\ln x = 4.8344$
14. $\ln x = 3.2752$ **15.** $\ln x = -0.3185$
16. $\ln x = -2.4316$ **17.** $\ln x = -1.5372$

18. Show that the relation between logarithms to the base b and logarithms to the base 10 is given by the equation

$$\log_b N = \frac{\log N}{\log b}\cdot$$

Use common logarithms and the equation in the preceding problem to find the approximate value of each of the following logarithms:

19. $\log_2 7$ **20.** $\log_5 9$ **21.** $\log_3 76$
22. $\log_9 105$ **23.** $\log_{0.4} 0.9$ **24.** $\log_{0.3} 0.5$

Evaluate each of the following expressions:

25. $\ln 32 - \ln \frac{1}{4} - \ln 16$ **26.** $\ln 48 + \ln \frac{1}{8} - \ln 12$

27. $3[\ln(4 - \sqrt{15}) - \ln(4 + \sqrt{15})]$

28. $10[\ln(\sqrt{7} + \sqrt{2}) - \ln(\sqrt{7} - \sqrt{2})]$

29. $\dfrac{e^v - e^{-v}}{2}$ if $v = 1.34$

30. $\dfrac{e^x + e^{-x}}{2}$ if $x = 1.67$

31. A formula for the charge on a condenser is $Q = CE(1 - e^{-(t/CR)})$, where Q is the charge in coulombs, C is the capacity of the condenser in farads,

E is the applied voltage, R is the resistance in ohms in series with the condenser, and t is the time in seconds after the voltage is applied to the circuit. Find Q if $C = 8.00 \times 10^{-5}f$, $R = 500$ ohms, $E = 120v$, and $t = 0.0300$ sec.

32. The atmospheric pressure p, in pounds per square inch, at an altitude h, in feet above sea level, is given by the formula $p = p_0e^{-kh}$, where p_0 is the pressure at sea level, k is a constant, and e is the base of the natural system of logarithms. If $p_0 = 14.72$ and the pressure is 13.68 lb per sq in. at an altitude of 2000 ft, find the value of k correct to three significant digits.

33. The equation of the probability curve is $y = ae^{-bx^2}$. Find the value of b if $a = 2$ when $x = 0.6$ and $y = 0.346$.

79. Logarithmic Solution of Equations

The use of logarithms not only simplifies the solution of certain types of equations, but also makes it possible to solve equations which cannot be solved by any other methods studied in algebra. The following examples illustrate the ideas that must be used here.

EXAMPLE 1. There are many places in chemistry, physics, and engineering where equations of the type $v^{1.41} = 32.3$ must be solved. Such equations occur particularly in the important science of thermodynamics. Solve this equation for v.

Solution: We take common logarithms of both sides of the equation to get

$$1.41 \log v = \log 32.3.$$

Therefore,
$$\log v = \frac{\log 32.3}{1.41} = \frac{1.50920}{1.41} = 1.0704,$$

and
$$v = 11.76. \quad \textit{Ans.}$$

EXAMPLE 2. If a sum of money, say P dollars, is invested at a rate of interest r, compounded annually for n years, the total accumulation A is given by the formula $A = P(1 + r)^n$. If the rate of interest is 3 per cent, find approximately how long it would take for P dollars to be doubled.

Solution: Under the conditions of the problem,

$$A = 2P,$$

and
$$r = 0.03,$$

so that we have to solve the following equation for n:

$$2P = P(1.03)^n,$$

or
$$(1.03)^n = 2.$$

Upon taking logarithms of both members of the last equation, we obtain
$$n \log 1.03 = \log 2,$$

or
$$n = \frac{\log 2}{\log 1.03}$$
$$= \frac{0.30103}{0.01284}$$
$$= 23.44.$$

Ordinarily, interest is not compounded for a fraction of an interest period; the compound interest formula is used for the number of entire periods, and the final amount is then computed at simple interest for the remaining fraction of a period. Hence, it is sufficient here to give as the result 23.4 yr approximately. *Ans.*

An equation in which the unknown occurs in an exponent is called an **exponential equation.** The preceding problem involved the solution of an exponential equation.

EXAMPLE 3. Solve the exponential equation $2^x = 3^{x-1}$.

Solution: Take logarithms of both sides to get
$$x \log 2 = (x - 1) \log 3.$$

Solve for x
$$x = \frac{\log 3}{\log 3 - \log 2} = \frac{0.47712}{0.47712 - 0.30103}$$
$$= \frac{0.47712}{0.17609} = 2.7095. \quad \textit{Ans.}$$

If logarithms of expressions in the unknown are involved in an equation, it is called a **logarithmic equation.**

EXAMPLE 4. Find N if $\log N = 0.724 + \log (N - 3)$.

Solution: From the tables, we find $0.724 = \log 5.2966$. Therefore, the given equation may be written
$$\log N = \log 5.2966 + \log (N - 3),$$

or
$$\log N = \log [5.2966(N - 3)]$$
$$= \log (5.2966N - 15.890).$$

If a and b are positive numbers, it follows from the equation $\log a = \log b$ that $a = b$. Hence, if N and $5.2966N - 15.890$ are positive numbers, we have

$$N = 5.2966N - 15.890,$$

or

$$4.2966N = 15.890,$$

and

$$N = 3.6982. \quad \textit{Ans.}$$

EXERCISES 66

Solve each of Equations 1 to 20 for the approximate value of the unknown letter.

1. $3^x = 7$

2. $5^{y-1} = 3^y$

3. $3^{s-1} \cdot 2^{3s} = 11$

4. $7^{3z} \cdot 5^{z-1} = 35$

5. $9^{2-3x} = 12^x$

6. $4^{-w} = 6^{w+2}$

7. $7^{2/v} = 61$

8. $11^{-1/x} = 42$

9. $5^{y+1} \cdot 3^{2y+3} = 8^{3y-4}$

10. $15^{2y} \cdot 2^{1-5y} = 58$

11. $(0.3)^z = 0.7$

12. $(0.04)^{2x} = 0.008$

13. $2.4 = \ln x^4$

14. $8 = \log v^{10}$

15. $\ln (2z + 13) = 4$

16. $\ln (4x - 15) = 3$

17. $\log w - \log (w + 4) = -1.23518$

18. $\log (x - 6) - \log (x + 6) = -0.92304$

19. $\ln (y + 10) - \ln (y - 10) = 0.60000$

20. $\ln (5x - 2) - \ln (x - 1) = 0.30000$

In each of the next two problems, find an approximate solution of the system of equations.

21. $2^{x+2y} = 32$
$3^{x-y} = 12$

22. $9^{x+y} = 4^{3x-1}$
$8^x = 3^y$

23. How many digits are there in 5^{20}?

24. How many digits are there in 12^{24}?

In Exercises 25 to 30 solve the equation for the letter which appears in each case after the semicolon.

25. $3 \ln y = 2 \ln x + \ln c; \quad y$

26. $4 \ln y = 2x + \ln c; \quad y$

27. $y = \dfrac{e^{2x} + e^{-2x}}{2}; \quad x$

(HINT: Solve first for e^x.)

28. $r = \dfrac{e^s - e^{-s}}{2}; \quad s$

(HINT: Solve first for e^s.)

29. $I = \dfrac{E}{R}(1 - e^{-(Rt/L)}); \quad t$

30. $Q = CE(1 - e^{(-t/CR)}); \quad R$

Chapter 13

THE BINOMIAL THEOREM AND MATHEMATICAL INDUCTION

80. The Binomial Theorem for Integral Exponents

From our previous work in algebra, we may list the following results:

$$(a + b)^1 = a + b; \tag{1}$$

$$(a + b)^2 = a^2 + 2ab + b^2; \tag{2}$$

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3. \tag{3}$$

Furthermore, the multiplication of both sides of (3) by $a + b$ gives

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4. \tag{4}$$

These four identities are special cases of a general formula for the expansion of $(a + b)^n$. Before attempting to write this formula, we may note the characteristics which appear to be common to the four special cases. If $n = 1, 2, 3$, or 4,

(1) The first term is a^n and the last term is b^n.

(2) The exponents of a decrease by 1 from term to term; the exponents of b increase by 1 from term to term; and the degree of each term in the two letters a and b is n.

(3) If the coefficient of any term is multiplied by the exponent of a in this term and divided by the number of the term, the coefficient of the next term is obtained.

For verification of (3), we find that the second term in each case has the coefficient $\frac{n}{1}$, that is, the exponent of a in the first term divided by 1,

the number of the first term. Similarly, the third term in (2), (3), and (4) has the coefficient $\dfrac{n(n-1)}{1 \cdot 2}$, which is

$$\frac{2(2-1)}{1 \cdot 2} = 1, \quad \text{for} \quad n = 2;$$

$$\frac{3(3-1)}{1 \cdot 2} = 3, \quad \text{for} \quad n = 3;$$

and
$$\frac{4(4-1)}{1 \cdot 2} = 6, \quad \text{for} \quad n = 4.$$

Assuming that the same characteristics are maintained for all positive integral values of n, we should have

$$(a + b)^n = a^n + \frac{n}{1} a^{n-1}b + \frac{n(n-1)}{1 \cdot 2} a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} a^{n-3}b^3 + \cdots + b^n. \quad (5)$$

This identity is called the **binomial formula.** The **binomial theorem** is the statement that the formula holds for all positive integral values of n. For the present, we shall assume the validity of this theorem; the proof is postponed to the end of this chapter.

In addition to the statements in (1), (2), and (3) above, it may be noted that

(4) The expansion of $(a + b)^n$ has $n + 1$ terms.

(5) The coefficients are symmetric, that is, the same coefficients are obtained counting from the left end as from the right.

EXAMPLE 1. Expand $(a + b)^5$.

Solution: By the binomial formula,

$$(a + b)^5 = a^5 + \frac{5}{1} a^4b + \frac{5 \cdot 4}{1 \cdot 2} a^3b^2 + \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3} a^2b^3 + \frac{5 \cdot 4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4} ab^4 + \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} b^5$$

$$= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5. \quad \textit{Ans.}$$

EXAMPLE 2. Expand $\left(\dfrac{x^2}{2} - 2y\right)^4$.

Solution: Let $a = \dfrac{x^2}{2}$ and $b = -2y$ in the binomial formula; then

$$\left(\frac{x^2}{2} - 2y\right)^4 = \left(\frac{x^2}{2}\right)^4 + \frac{4}{1}\left(\frac{x^2}{2}\right)^3(-2y) + \frac{4 \cdot 3}{1 \cdot 2}\left(\frac{x^2}{2}\right)^2(-2y)^2$$

$$+ \frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3}\left(\frac{x^2}{2}\right)(-2y)^3 + \frac{4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4}(-2y)^4$$

$$= \frac{x^8}{16} - x^6y + 6x^4y^2 - 16x^2y^3 + 16y^4. \quad \textit{Ans.}$$

In order to avoid confusion, the student should make no simplifications until he has written down the expansion completely.

EXAMPLE 3. Use the binomial formula to find the value of $(1.01)^{10}$ correct to five significant digits.

Solution: We write $(1.01)^{10} = (1 + 0.01)^{10}$ and make $a = 1$, $b = 0.01$, and $n = 10$ in the binomial formula to obtain

$$(1.01)^{10} = 1 + \frac{10}{1}(0.01) + \frac{10 \cdot 9}{1 \cdot 2}(0.01)^2$$

$$+ \frac{10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3}(0.01)^3 + \cdots + (0.01)^{10}$$

$$= 1 + 0.1 + 0.0045 + 0.000120 + \cdots$$

$$= 1.1046, \text{ correct to five significant digits.} \quad \textit{Ans.}$$

The student may verify the fact that the terms omitted in this calculation do not affect the accuracy of the result as given.

EXERCISES 67

Expand the following expressions by the binomial formula:

1. $(x + 3)^4$ **2.** $(w - 2)^4$ **3.** $(a^2 - 2b)^4$
4. $(3x^2 + 2y^3)^4$ **5.** $(2d^3 - 3c^5)^4$ **6.** $(5u^{-2} + 4v^{-2})^4$
7. $(a^{-2} - b^{-2})^4$ **8.** $(3 - y^2)^5$ **9.** $(x^2 + y)^5$

10. $(s^2 + 2t^2)^5$ **11.** $\left(2h + \frac{k}{3}\right)^4$ **12.** $\left(c\sqrt{3} - \frac{d}{2}\right)^4$

13. $(u^2 - 2)^6$ **14.** $(r^2 - y^3)^6$ **15.** $\left(\frac{a^3}{b} - \frac{b}{a^3}\right)^4$

16. $(x + y)^7$ **17.** $(2 - f^2)^7$ **18.** $(x^{-4} - y^{\frac{1}{2}})^6$

19. $\left(\frac{e}{3} + m\sqrt{5}\right)^6$ **20.** $\left(4z^5 - \frac{x^2y^3}{z}\right)^5$ **21.** $\left(2a^2b^3 - \frac{c}{2a}\right)^7$

22. $(\sqrt{x} - \sqrt{y})^8$ **23.** $(u^2 - u + 1)^4$ **24.** $(x^2 + 2x - 1)^4$

Use the binomial formula to find the value of the following expressions correct to five significant digits:

25. $(1.02)^6$ **26.** $(1.01)^8$ **27.** $(1.03)^9$

28. $(1.04)^9$ **29.** $(0.98)^6$ **30.** $(0.99)^5$

81. The Coefficients in the Binomial Formula

In obtaining the expansion of $(a + b)^4$ from that of $(a + b)^3$ in the preceding section, we multiplied by $a + b$. It is instructive to repeat this multiplication, paying particular attention to the formation of the coefficients.

$$
\begin{array}{lclclcr}
a^3 & + & 3a^2b & + & 3ab^2 & + & b^3 \\
 & & & & & & a + b \\
\hline
a^4 & + & 3a^3b & + & 3a^2b^2 & + & ab^3 \\
 & & a^3b & + & 3a^2b^2 & + & 3ab^3 + b^4 \\
\hline
\multicolumn{7}{l}{a^4 + (3 + 1)a^3b + (3 + 3)a^2b^2 + (1 + 3)ab^3 + b^4}
\end{array}
$$

If we write the coefficients in the first and last lines of the multiplication as follows:

$$
\begin{array}{ccccccccc}
 & 1 & & 3 & & 3 & & 1 & \\
\swarrow & & \searrow\ \swarrow & & \searrow\ \swarrow & & \searrow\ \swarrow & & \searrow \\
1 & & 1 + 3 & & 3 + 3 & & 3 + 1 & & 1
\end{array}
$$

it becomes clear how the binomial expansion for any integral exponent may be obtained from the expansion for the preceding integral exponent. For example, from the coefficients for $n = 4$, the coefficients for $n = 5$ may be obtained; thus,

$$
\begin{array}{ccccccccccc}
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
\swarrow & & \searrow\ \swarrow & & \searrow\ \swarrow & & \searrow\ \swarrow & & \searrow\ \swarrow & & \searrow \\
1 & & 1 + 4 & & 4 + 6 & & 6 + 4 & & 4 + 1 & & 1
\end{array}
$$

This discussion explains the arrangement of coefficients in the following array, which is called **Pascal's triangle:**

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & \mathbf{1} & & \mathbf{1} & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & \mathbf{1} & & \mathbf{3} & & \mathbf{3} & & \mathbf{1} & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & \mathbf{1} & & \mathbf{5} & & \mathbf{10} & & \mathbf{10} & & \mathbf{5} & & \mathbf{1} & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
\cdot & & \cdot & & \cdot & & \cdot & & \cdot & & \cdot & & \cdot & & \cdot
\end{array}
$$

The boldface numbers in each horizontal line indicate the exponent to which the coefficients in that line belong. The 1 at the top for $(a + b)^0$ has been added for the sake of symmetry. If the exponent n is not too large, the Pascal triangle furnishes a simple scheme for finding the coefficients. In the proof of the bionomial theorem, we shall make use of the rule by which one line of the array is formed from the preceding one.

We shall need the formula for the general term of the binomial expansion. This term may be obtained from inspection of Formula (5) in the preceding section. For any term after the first, say term number $k + 1$, we should have

$$\frac{\overbrace{n(n-1)(n-2)\cdots(n-k+1)}^{k \text{ factors}}}{1\cdot 2\cdot 3\cdot \cdots \cdot k}\, a^{n-k}b^k.$$

If k is a positive integer, the product

$$1\cdot 2\cdot 3\cdot \cdots \cdot (k-1)k$$

is symbolized by $k!$, and is read "factorial k." For example,

$$5! = 1\cdot 2\cdot 3\cdot 4\cdot 5.$$

The general binomial coefficient is often indicated by the symbol $\binom{n}{k}$, defined by the equations

$$\binom{n}{0} = 1,$$

and $\displaystyle\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$, $k = 1, 2, 3, \cdots.$

In terms of this definition, we may write

$$(a+b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \cdots$$
$$+ \binom{n}{k} a^{n-k}b^k + \cdots + \binom{n}{n} b^n. \qquad (7)$$

The student should notice that the *number* of the general term has been taken as $k + 1$.

EXAMPLE 1. Write and simplify the sixth term of the expansion of $(5x - \frac{1}{10}y)^{12}$.

Solution: $k + 1 = 6$; $k = 5$. Hence, the sixth term is

$$\binom{12}{5} (5x)^7 \left(-\frac{1}{10}y\right)^5 = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} (5x)^7 \left(-\frac{1}{10}y\right)^5$$
$$= -\frac{2475}{4} x^7y^5. \quad \textit{Ans.}$$

EXAMPLE 2. Find the term which involves no letter in the expansion of $\left(c^2 - \frac{2}{c^3}\right)^{10}$.

Solution: The general term of the expansion of the given binomial is

$$\binom{10}{k} (c^2)^{10-k} \left(-\frac{2}{c^3}\right)^k.$$

The power of c in this expression is

$$\frac{c^{20-2k}}{c^{3k}} = c^{20-5k}.$$

If no letter is to be involved, we must have

$$20 - 5k = 0,$$

or

$$k = 4.$$

The required term is the 5th term, namely,

$$\frac{10 \cdot 9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4} (c^2)^6 \left(-\frac{2}{c^3}\right)^4 = 3360. \quad \textit{Ans.}$$

EXERCISES 68

In Examples 1 to 6 the indicated term in each expansion is to be found without writing the preceding terms.

1. $(y - 2)^{13}$; 7th term
2. $(z^2 + 4)^{10}$; 6th term
3. $(u^3 + v^4)^{12}$; 8th term
4. $(2a^{-4} - b^{-2})^{11}$; 7th term
5. $\left(\dfrac{x}{y} + \dfrac{z^2}{c}\right)^{14}$; 9th term
6. $\left(\dfrac{2u}{v} - \dfrac{v^2}{2u}\right)^{13}$; 10th term
7. Find the coefficient of z^{14} in $\left(z^3 + \dfrac{2}{z^2}\right)^{13}$.
8. Find the coefficient of y^{-8} in $\left(y - \dfrac{1}{y^4}\right)^{12}$.
9. Find the middle term of $\left(\dfrac{3}{x^2} - \dfrac{x^5}{9}\right)^{12}$.
10. Find the middle term of $\left(\dfrac{c^2}{2} - b^{-3}\right)^{10}$.
11. Find the term which involves y^{-8} in $\left(\dfrac{1}{4y^4} - \dfrac{2y^2}{3}\right)^{14}$.
12. Find the term which involves b^{-10} in $\left(6b^5 - \dfrac{1}{3b^8}\right)^{11}$.
13. Find the term which involves no x in $\left(\dfrac{1}{2x^4} + 4x^2\right)^{12}$.
14. Find the term which involves no z in $\left(\dfrac{1}{z} + z^4\right)^{20}$.
15. Simplify: $\dfrac{8!}{6!}$; $\dfrac{11!}{7!}$; $\dfrac{(5!)(7!)}{(10!)(3!)}$.
16. Simplify: $\dfrac{(n+2)!}{n!}$; $\dfrac{(k-3)!}{k!}$; $\dfrac{(s-3)!}{(s-1)!}$.

82. The Binomial Series

If $a = 1$ and $b = x$ in the binomial formula, we have

$$(1 + x)^n = 1 + nx + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots + \binom{n}{k}x^k + \cdots. \tag{8}$$

In the case where n is a positive integer (or zero), Equation (8) will be the binomial formula for the special values of a and b; the expression on the right will terminate after $n + 1$ terms, the last one being x^n. This appears by inspection of the formula

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}.$$

In this formula, k is a positive integer. Hence, the value zero can be obtained only if n is a positive integer.

If n is not a positive integer or zero, the expansion may be continued without end. The result so considered is called the **binomial series.** It is shown in calculus that *if* x *is numerically less than* 1, the binomial series represents $(1 + x)^n$ in the following sense: The sum of the first s terms of the series may be made to approximate $(1 + x)^n$ with any desired degree of accuracy by taking s sufficiently large. That is, if x has any value between 1 and -1, we may find the value of $(1 + x)^n$ as accurately as we please by taking enough consecutive terms of the series.

EXAMPLE 1. Find the first four terms of the binomial series for $\dfrac{1}{\sqrt{a^2 - x^2}}\cdot$ Assume that a is positive and that $x^2 < a^2$.

Solution:
$$\frac{1}{\sqrt{a^2 - x^2}} = \frac{1}{a\sqrt{1 - \dfrac{x^2}{a^2}}} = \frac{1}{a}\left(1 - \frac{x^2}{a^2}\right)^{-\frac{1}{2}}$$

$$= \frac{1}{a}\left[1 + \frac{-\frac{1}{2}}{1}\left(-\frac{x^2}{a^2}\right) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{1\cdot 2}\left(-\frac{x^2}{a^2}\right)^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{1\cdot 2\cdot 3}\left(-\frac{x^2}{a^2}\right)^3 + \cdots\right]$$

$$= \frac{1}{a}\left(1 + \frac{x^2}{2a^2} + \frac{3x^4}{8a^4} + \frac{5x^6}{16a^6} + \cdots\right)$$

$$= \frac{1}{a} + \frac{x^2}{2a^3} + \frac{3x^4}{8a^5} + \frac{5x^6}{16a^7} + \cdots. \quad \textit{Ans.}$$

EXAMPLE 2. Find the value of $(1.02)^{-1}$ correct to four decimal places.

Solution: The binomial series for $n = -1$ is given by

$$(1 + x)^{-1} = 1 + \frac{-1}{1!}x + \frac{(-1)(-2)}{2!}x^2 + \frac{(-1)(-2)(-3)}{3!}x^3 + \cdots + \frac{(-1)(-2)\cdots(-1 - k + 1)}{k!}x^k + \cdots$$

$$= 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots.$$

We may now replace x by 0.02 to obtain

$$(1.02)^{-1} = 1 - 0.02 + 0.0004 - 0.000\,008 + \cdots$$
$$= 0.9804, \text{ correct to four decimal places.} \quad \textit{Ans.}$$

It is beyond the scope of this book to treat the error committed by breaking off the binomial series as we broke it off in the example above. In the exercises that follow, where the values of x to be used are small, calculations should be carried to at least one place beyond the number required, the final answer being obtained by rounding off.

EXAMPLE 3. Find $\sqrt[4]{620}$ correct to five decimal places by using the binomial series.

Solution:
$$\sqrt[4]{620} = (625 - 5)^{1/4} = [625(1 - \tfrac{1}{125})]^{1/4}$$
$$= 5(1 - 0.008)^{1/4}$$
$$= 5\left[1 + \frac{1}{4}(-0.008) + \frac{(\frac{1}{4})(-\frac{3}{4})}{1 \cdot 2}(-0.008)^2 + \cdots\right]$$
$$= 5(1 - 0.002 - 0.000\,006 - \cdots)$$
$$= 5(0.997\,994) = 4.98997. \quad \textit{Ans.}$$

EXERCISES 69

In each of Exercises 1 to 8, first factor the given binomial to obtain a binomial of the form $(1 + x)^n$, where $|x| < 1$; then expand to four terms by employing the binomial series, and simplify. (Compare with Example 1 of the preceding discussion.)

1. $(1 + 2x)^{-3}$, $|2x| < 1$

2. $(3 - 4y)^{-2}$, $|4y| < 3$

3. $(6 + 5z)^{-1}$, $|5z| < 6$

4. $(1 - 3w)^{-1/2}$, $|3w| < 1$

5. $(u^{-4} + v^{-4})^{1/2}$, $u^{-4} > v^{-4}$

6. $(x^{-3} + y^{-3})^{2/3}$, $|x^{-3}| > |y^{-3}|$

7. $\left(\dfrac{a^4}{16} - b^{-6}\right)^{-1/2}$, $a^4 > 16b^{-6}$

8. $\left(\dfrac{z^6}{27} - 1\right)^{-1/3}$, $z^6 > 27$

Use the binomial series to find the value of each of the following radicals correct to four significant digits:

9. $\sqrt{1.04}$ **10.** $\sqrt{1.2}$ **11.** $\sqrt{0.97}$

12. $\sqrt{37}$ **13.** $\sqrt{123}$ **14.** $\sqrt{79}$

15. $\sqrt[3]{9}$ **16.** $\sqrt[3]{26}$ **17.** $\sqrt[5]{33}$

18. $\dfrac{1}{\sqrt[4]{18}}$ **19.** $\dfrac{1}{\sqrt[3]{997}}$ **20.** $\dfrac{2}{\sqrt[5]{239}}$

21. For small values of d, the difference $\sqrt{1+d} - \sqrt{1-d}$ is given approximately by the expression $d + \frac{1}{8}d^3$. Obtain this result by means of the binomial series and check for $d = 0.1$.

22. For sufficiently small values of the ratio $|h/x|$, the difference $\dfrac{1}{\sqrt[3]{x}} - \dfrac{1}{\sqrt[3]{x+h}}$ may be approximated by $\dfrac{h}{3x^{4/3}}$. Obtain this approximation by means of the binomial series and check for $x = 27$, $h = 1$.

83. Mathematical Induction

We shall illustrate the type of reasoning called **mathematical induction** by proving the compound-interest formula

$$A = P(1+r)^n.$$

This formula gives the amount A to which an original principal P would accumulate in n years if invested at an interest rate r, compounded annually. The phrase "compounded annually" means that at the end of each year the interest is added to the principal and is itself allowed to earn interest.

During the first year the principal P earns interest Pr; so the new principal at the beginning of the second year is

$$P + Pr = P(1+r).$$

During the second year the new principal earns the interest

$$P(1+r)r.$$

Thus, the principal at the beginning of the third year is

$$P(1+r) + P(1+r)r = P(1+r)(1+r) = P(1+r)^2.$$

The student may carry out the same argument to show that at the beginning of the fourth year the total accumulation is

$$P(1+r)^3.$$

While the evidence presented here makes the given formula quite plausible, nothing has actually been proved beyond the fact that the formula is correct for $n = 1$, 2, and 3. There is always the possibility, for example, that a proposed formula may be valid for $n = 1, 2, 3, \cdots, 1000$ and then fail for $n = 1001$.

However, suppose that we could derive a type of **inheritance property**, namely, that the validity of the formula for any integral value, say $n = s$, carries with it the validity of the corresponding formula for the succeeding integral value $n = s + 1$. Then, we could argue that if

the formula has been verified for some integral value, say $n = 3$, the inheritance property makes the corresponding formula valid for $n = 4$. The validity for $n = 4$ carries over to validity for $n = 5$, and so on.

In order to be able to apply this argument to the proposed compound-interest formula, *we assume that the formula is valid for* n = s, that is, we assume that the total accumulation at the end of s years is

$$P(1 + r)^s.$$

Now, during the $(s + 1)$th year, the interest earned would be $P(1 + r)^s r$. The total accumulation at the end of this year would be

$$\begin{aligned} P(1 + r)^s + P(1 + r)^s r &= P(1 + r)^s(1 + r) \\ &= P(1 + r)^{s+1}. \end{aligned}$$

This result shows the existence of the inheritance property; this formula is the one we would obtain if $s + 1$ were put in place of n in the proposed formula. In other words, *if* the proposed formula is valid for $n = s$, it must be valid for $n = s + 1$.

The argument may be completed as follows. The formula $A = P(1 + r)^n$ has been verified for $n = 1$, 2, and 3. By the inheritance property, it must be correct for $n = 4$. Since it is correct for $n = 4$, it must be valid for $n = 5$, and so on for any positive integral value of n.

Mathematical induction is used frequently in proving the validity of a proposed formula or theorem which involves positive integers. We may summarize the steps in the reasoning as follows:

(1) The proposition is verified for some specific value of n (usually $n = 1$).

(2) The inheritance property is shown. This means that from the *assumption* of the validity of the proposition for $n = s$, the validity for $n = s + 1$ is *derived.*

(3) The preceding steps are recapitulated in the statement that since the proposition holds for $n = 1$, the inheritance property makes it hold for $n = 2$; then, since it holds for $n = 2$, it must hold for $n = 3$; and so on. This statement is not logically necessary but is inserted to round out the discussion and to help the student make certain that he has not omitted either step (1) or step (2).

Example 1. Prove that the sum of the first n even integers is $n(n + 1)$, that is,

$$2 + 4 + 6 + \cdots + 2n = n(n + 1).$$

Solution—Step (1): If $n = 1$, the formula is easily verified, for the left side becomes 2 and the right side becomes $1(2) = 2$.

Step (2): *Assume* that for some integral value of n, say s, we have

$$2 + 4 + 6 + \cdots + 2s = s(s + 1).$$

Now, add $2s + 2$, the even integer following $2s$, to both sides.

$$\begin{aligned} 2 + 4 + 6 + \cdots + 2s + (2s + 2) &= s(s + 1) + (2s + 2) \\ &= s(s + 1) + 2(s + 1) \\ &= (s + 1)(s + 2). \end{aligned}$$

This is the proposed formula with n replaced by $s + 1$; hence, the inheritance property is established.

Step (3): The formula has been verified for $n = 1$. By step (2), it must hold for $n = 2$. Again, by step (2), it must be valid for $n = 3$, and so on to any desired positive integral value of n.

EXAMPLE 2. Prove that $x^{2n-1} + y^{2n-1}$ is exactly divisible by $x + y$ for any positive integral value of n.

Solution—Step (1): If $n = 1$, we have

$$x^{2n-1} + y^{2n-1} = x + y,$$

which is obviously exactly divisible by $x + y$. If $n = 2$, we have $x^3 + y^3$, which is also exactly divisible by $x + y$.

Step (2): *Assume* that the proposition is correct for integral values s and $s - 1$. This means that we are assuming

$$x^{2s-1} + y^{2s-1} = (x + y)Q_1, \tag{1}$$

and

$$x^{2s-3} + y^{2s-3} = (x + y)Q_2, \tag{2}$$

where Q_1 and Q_2 are rational integral expressions. Let us now try to operate on these equations and combine them in such a manner that we obtain an expression for $x^{2s+1} + y^{2s+1}$.

First, multiply both members of Equation (1) by x^2. Then

$$x^{2s+1} + x^2y^{2s-1} = x^2(x + y)Q_1.$$

Then, multiply both sides of the same equation by y^2.

$$y^2x^{2s-1} + y^{2s+1} = y^2(x + y)Q_1.$$

Add the last two equations.

$$\begin{aligned} x^{2s+1} + x^2y^{2s-1} + y^2x^{2s-1} + y^{2s+1} &= x^2(x + y)Q_1 + y^2(x + y)Q_1, \\ &= (x + y)(x^2 + y^2)Q_1. \end{aligned}$$

Subtract $x^2y^{2s-1} + y^2x^{2s-1}$ from both sides.

$$x^{2s+1} + y^{2s+1} = (x + y)(x^2 + y^2)Q_1 - x^{2s-1}y^2 - x^2y^{2s-1}$$
$$= (x + y)(x^2 + y^2)Q_1 - x^2y^2(x^{2s-3} + y^{2s-3}).$$

Since by Equation (2) the expression within the last parentheses is equal to $(x + y)Q_2$, we have

$$x^{2s+1} + y^{2s+1} = (x + y)(x^2 + y^2)Q_1 - x^2y^2(x + y)Q_2$$
$$= (x + y)[(x^2 + y^2)Q_1 - x^2y^2Q_2].$$

Because of our assumptions, the expression in brackets must be a rational integral one. Hence, this conclusion indicates that $x^{2s+1} + y^{2s+1}$ is divisible by $x + y$ if $x^{2s-1} + y^{2s-1}$ and $x^{2s-3} + y^{2s-3}$ are divisible by $x + y$. This establishes the inheritance property.

Step (3): The proposition has been verified for $n = 1$ and 2. By step (2), it must hold for $n = 3$. Since it holds for $n = 2$ and 3, it must hold for $n = 4$, and so on.

NOTE: The slight change in the form of the assumption in this example constitutes no change in the fundamental reasoning; in fact, we may start by assuming the proposed formula to be valid for all integral values of n from 1 to s, inclusive, without changing the essential ideas of a proof by mathematical induction.

COMMENTS: The importance of carrying out both steps (1) and (2) may be emphasized by the following discussion: Step (2) may easily be carried out with the *false* proposition that $n^2 + n$ is always an odd integer. If it is assumed that the proposition is valid, we have, on replacing n by $n + 1$,

$$(n + 1)^2 + (n + 1) = n^2 + 2n + 1 + n + 1$$
$$= (n^2 + n) + 2n + 2.$$

Since $2n + 2$ is always even, the expression $(n^2 + n) + 2n + 2$ is odd *if* $n^2 + n$ is odd. (The sum of an odd and an even integer is always odd.) However, $n^2 + n = n(n + 1)$ is always even; it is the product of an odd and an even integer. Hence, the original proposition is patently false. This illustration shows that step (2) alone is not enough.

Another simple illustration shows that step (1) alone is also not sufficient. Consider the proposed formula

$$1 + 3 + 5 + \cdots + (2n - 1) = n^3 - 5n^2 + 11n - 6.$$

We find by substitution that the formula gives correct results for $n = 1$, 2, and 3. This verification might tempt us to assume the validity of

the formula for all integral values of n. However, step (2) of the mathematical induction proof cannot be carried through. For the formula is wrong for every value of n except $n = 1$, 2, and 3. For instance, putting $n = 4$, we find on the left side

$$1 + 3 + 5 + 7 = 16;$$

whereas on the right side

$$4^3 - 5(4^2) + 11(4) - 6 = 22.$$

EXERCISES 70

Prove each of the following statements by mathematical induction. The letter n stands for a positive integer throughout.

1. $1 + 2 + 3 + \cdots + n = \frac{n}{2}(n + 1)$

2. $3 + 7 + 11 + \cdots + (4n - 1) = n(2n + 1)$

3. $6k + 12k + 18k + \cdots + 6nk = 3n(n + 1)k$

4. $r + (r + 2) + (r + 4) + \cdots + (r + 2n - 2) = n(r + n - 1)$

5. $1 + 3 + 5 + \cdots + (2n - 1) = n^2$

6. $1 + 5 + 9 + \cdots + (4n - 3) = n(2n - 1)$

7. $4 + 4^2 + 4^3 + \cdots + 4^n = \dfrac{4(4^n - 1)}{3}$

8. $b + b^2 + b^3 + \cdots + b^n = \dfrac{b(b^n - 1)}{b - 1}$, $b \neq 1$

9. $2 \cdot 4 + 4 \cdot 6 + 6 \cdot 8 + \cdots + 2n(2n + 2) = \frac{4n}{3}(n + 1)(n + 2)$

10. $3 \cdot 3 + 6 \cdot 4 + 9 \cdot 5 + \cdots + 3n(n + 2) = \frac{n}{2}(n + 1)(2n + 7)$

11. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n}{6}(n + 1)(2n + 1)$

12. $2 \cdot 5 + 5 \cdot 8 + 8 \cdot 11 + \cdots + (3n - 1)(3n + 2) = n(3n^2 + 6n + 1)$

13. $1 \cdot 6 + 4 \cdot 9 + 7 \cdot 12 + \cdots + (3n - 2)(3n + 3) = 3n(n^2 + 2n - 1)$

14. $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \cdots + \dfrac{1}{n(n + 1)} = \dfrac{n}{n + 1}$

15. $\dfrac{1}{1 \cdot 3} + \dfrac{1}{3 \cdot 5} + \dfrac{1}{5 \cdot 7} + \cdots + \dfrac{1}{(2n - 1)(2n + 1)} = \dfrac{n}{2n + 1}$

16. $\dfrac{1}{1 \cdot 5} + \dfrac{1}{5 \cdot 9} + \dfrac{1}{9 \cdot 13} + \cdots + \dfrac{1}{(4n - 3)(4n + 1)} = \dfrac{n}{4n + 1}$

17. $1 + 2 \cdot 2 + 3 \cdot 2^2 + 4 \cdot 2^3 + \cdots + n2^{n-1} = 1 + (n - 1)2^n$

18. $1 + 2 \cdot 3 + 3 \cdot 3^2 + 4 \cdot 3^3 + \cdots + n3^{n-1} = \dfrac{1 + (2n - 1)3^n}{4}$

19. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4} = (1 + 2 + 3 + \cdots + n)^2$

20. $1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3 = n^2(2n^2 - 1)$

21. $x - y$ is a factor of $x^n - y^n$

22. $x + y$ is a factor of $x^{2n} - y^{2n}$

23. Show that the formula

$$1 + 2 + 3 + \cdots + n$$
$$= \frac{n}{2}(n+1) + (n-1)(n-2)(n-3)\cdots(n-1000)$$

is valid for the first thousand values of n but is wrong for any other value of n. (Hint: Use the result of Problem 1.)

84. The Proof of the Binomial Formula

At the beginning of this chapter, we verified the binomial formula for $n = 1, 2, 3$, and 4. We now *assume* that the formula is correct for some integral value $n = s$, that is,

$$(a+b)^s = \binom{s}{0}a^s + \binom{s}{1}a^{s-1}b + \binom{s}{2}a^{s-2}b^2 + \cdots$$
$$+ \binom{s}{k-1}a^{s-k+1}b^{k-1} + \binom{s}{k}a^{s-k}b^k + \cdots + \binom{s}{s}b^s.$$

Multiplying both sides of this equation by $a + b$, we obtain

$$(a+b)^{s+1} = \binom{s}{0}a^{s+1} + \left[\binom{s}{0} + \binom{s}{1}\right]a^sb + \left[\binom{s}{1} + \binom{s}{2}\right]a^{s-1}b^2 + \cdots$$
$$+ \left[\binom{s}{k-1} + \binom{s}{k}\right]a^{s-k+1}b^k + \cdots + \binom{s}{s}b^{s+1}.$$

(Recall the scheme for obtaining Pascal's triangle.)

Now, $\qquad \binom{s}{0} = \binom{s+1}{0} = 1, \quad$ by definition;

also, $\qquad \binom{s}{s} = \dfrac{s!}{s!} = 1 \quad \text{and} \quad \binom{s+1}{s+1} = \dfrac{(s+1)!}{(s+1)!} = 1.$

Moreover,

$$\binom{s}{k-1} + \binom{s}{k} = \frac{s(s-1)\cdots(s-k+2)}{(k-1)!} + \frac{s(s-1)\cdots(s-k+1)}{k!}$$
$$= \frac{s(s-1)\cdots(s-k+2)}{(k-1)!}\left(1 + \frac{s-k+1}{k}\right)$$

$$= \frac{s(s-1)\cdots(s-k+2)}{(k-1)!} \cdot \frac{s+1}{k}$$

$$= \frac{(s+1)(s)(s-1)\cdots[(s+1)-k+1]}{k!}$$

$$= \binom{s+1}{k}.$$

Hence, if the binomial formula is correct for $n = s$, we must have

$$(a+b)^{s+1} = \binom{s+1}{0}a^{s+1} + \binom{s+1}{1}a^s b + \binom{s+1}{2}a^{s-1}b^2 + \cdots$$
$$+ \binom{s+1}{k}a^{s+1-k}b^k + \cdots + \binom{s+1}{s+1}b^{s+1},$$

which is the binomial formula for $n = s + 1$. This shows the inheritance property and completes the proof of the formula for positive integral exponents.

Chapter 14

PROGRESSIONS

85. Sequences

In algebra, we define a **sequence** as a set of numbers arranged according to some law that determines which number is first, which is second, etc. For instance, the digits 1, 2, 3, $\cdots$, 9 arranged in natural order furnish a simple example of a sequence. Other illustrations are

(1) The positive even integers from 2 to 10 arranged in order of increasing magnitude: 2, 4, 6, 8, 10.

(2) The reciprocals of the positive integers from 1 to 7 arranged in order of decreasing magnitude: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}$.

The numbers in a sequence are called **terms** and are spoken of as the *first term*, the *second term*, and so on, according to their placement in the sequence. We shall be concerned in this chapter with the properties of two very special types of sequences known as **progressions.**

86. Arithmetic Progressions

An **arithmetic progression** is a sequence in which each term after the first is formed by adding the same fixed number to the preceding term. The number which is added in this manner is called the **common difference.**

Illustrations: (*a*) The sequence 2, 4, 6, 8, 10 is an arithmetic progression in which each term after the first is formed by adding 2 to the preceding term.

(*b*) The sequence 0.3, 0.2, 0.1, 0.0, -0.1, -0.2 is an arithmetic progression in which the common difference is -0.1.

In order to describe an arithmetic progression completely, we specify

the following numbers:

a_1, the first term,

d, the common difference,

and n, the number of terms.

Thus, if $a_1 = 4$, $d = 3$, and $n = 5$, we have the progression

$$4, 7, 10, 13, 16.$$

Similarly, for $a_1 = \frac{2}{3}$, $d = -\frac{1}{3}$, and $n = 4$, we have the progression

$$\tfrac{2}{3}, \tfrac{1}{3}, 0, -\tfrac{1}{3}.$$

In general, the first n terms of an arithmetic progression may be written

$$a_1, a_1 + d, a_1 + 2d, a_1 + 3d, \cdots, a_1 + (n - 1)d.$$

Two other numbers of interest in the arithmetic progression are

a_n, the nth term,

and S_n, the sum of n terms.

The five numbers a_1, a_n, d, n, and S_n, often called the **elements** of the arithmetic progression, bear certain important relations to each other. For example, the formula

$$a_n = a_1 + (n - 1)d \tag{1}$$

is a relation which is obtained directly from the law of formation. This is seen from the fact that the second term is obtained by adding d to the first term; the third term by adding $2d$ to the first term, and so on; the number of times d is added to a_1 always being the number of the term diminished by 1.

The sum of n terms is by definition

$$S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + (a_n - 2d) + (a_n - d) + a_n,$$

where a few terms have been written from each end in an obvious fashion.

We may rearrange the right side of the preceding equation so that the terms are in reverse order, thus:

$$S_n = a_n + (a_n - d) + (a_n - 2d) + \cdots + (a_1 + 2d) + (a_1 + d) + a_1.$$

The addition of the members of the last two equations gives $2S_n$ on the left. On the right, the terms d, $2d$, and so on, will be annulled and

the quantity $a_1 + a_n$ will occur n times. (Why?) Therefore,

$$2S_n = n(a_1 + a_n),$$

or

$$S_n = n\left(\frac{a_1 + a_n}{2}\right). \tag{2}$$

Notice that the quantity $\frac{a_1 + a_n}{2}$ is the ordinary average of the first and nth terms.

Example 1. Find the twentieth term and the sum of the first twenty terms of an arithmetic progression whose first term is 1 and whose common difference is 9.

Solution: $a_1 = 1$, $d = 9$, and $n = 20$. Hence, by Equation (1),

$$\begin{aligned} a_{20} &= a_1 + (n - 1)d \\ &= 1 + (19)(9) = 172. \quad \textit{Ans.} \end{aligned}$$

Furthermore, by Equation (2), $S_{20} = n\left(\frac{a_1 + a_{20}}{2}\right)$

$$= 20\left(\frac{1 + 172}{2}\right) = 1730. \quad \textit{Ans.}$$

Example 2. For an arithmetic progression, the sum of the first ten terms is -3.875, and the tenth term is 0.625. Find the first term and the common difference.

Solution: From (2), we have

$$-3.875 = 10\left(\frac{a_1 + 0.625}{2}\right),$$

or

$$-0.775 = a_1 + 0.625,$$

and

$$a_1 = -1.4. \quad \textit{Ans.}$$

Also, by (1),

$$0.625 = -1.4 + 9d,$$

or

$$9d = 2.025,$$

and

$$d = 0.225. \quad \textit{Ans.}$$

Example 3. How many terms are there in the arithmetic progression for which $a_1 = 3$, $d = 5$, and $S_n = 255$?

Solution: From (2), we have

$$255 = n\left(\frac{3 + a_n}{2}\right),$$

or

$$510 = n(3 + a_n).$$

From (1),

$$a_n = 3 + (n - 1)5$$
$$= 5n - 2.$$

Hence,

$$510 = n(3 + 5n - 2),$$

and

$$5n^2 + n - 510 = 0.$$

Since n must be a positive integer if this problem has a solution, we may expect to solve this equation by factoring. Thus,

$$(5n + 51)(n - 10) = 0,$$
$$n = 10. \quad \textit{Ans.}$$

The terms of an arithmetic progression which occur between a_1 and a_n are called **arithmetic means.** The student may show easily that the arithmetic mean between two numbers is the ordinary average of the two numbers.

EXAMPLE 4. Insert five arithmetic means between 2 and -16.

Solution: Counting the given terms, we must set up the arithmetic progression whose first term is 2 and whose seventh term is -16. Therefore, using (1), we write

$$a_7 = a_1 + 6d,$$

that is,

$$-16 = 2 + 6d.$$

This equation gives $d = -3$, so that the required means are -1, -4, -7, -10, and -13. *Ans.*

EXERCISES 71

1. Determine which of the following sequences are arithmetic progressions, and find the common difference d for each such progression:

(*a*) 7, 12, 17, 22
(*b*) 67, 54, 41, 28
(*c*) 6.4, 5.6, 4.8, 4.2
(*d*) $9c^2$, $16c^2$, $23c^2$, $30c^2$
(*e*) $2 + 7\sqrt{5}$, $3\sqrt{5}$, $-2 - 7\sqrt{5}$
(*f*) $a^2 - 4b^2$, $2a^2 - b^2$, $3a^2 + 2b^2$
(*g*) $11e + 7f$, $7e + 2f$, $3e - 3f$
(*h*) $11k + 6m$, $6k + 13m$, $20k + 20m$

2. Find the thirteenth and twenty-fourth terms of the arithmetic progression $-13, -6, 1, 8, \cdots$.

3. Find the fifteenth and thirty-third terms of the arithmetic progression $19, 17\frac{1}{4}, 15\frac{1}{2}, 13\frac{3}{4}, \cdots$.

4. Find the eleventh and seventeenth terms of the arithmetic progression $a + 24e, 4a + 20e, 7a + 16e, \cdots$.

5. Find the tenth and sixteenth terms of the arithmetic progression $7v^2 - 4z, 2v^2 + 7z, -3v^2 + 18z, \cdots$.

6. Find S_{14} for the arithmetic progression

$$-13 + 4\sqrt{2},\ -9 + 7\sqrt{2},\ -5 + 10\sqrt{2}, \cdots.$$

7. Find S_9 for the arithmetic progression $\frac{6}{5c}, \frac{3a+1}{5c}, \frac{6a-4}{5c}, \cdots$.

In each of Problems 8 to 16 certain elements of an arithmetic progression are given. Find the indicated elements whose values are not given.

8. $a_1 = 5, d = 8;\ a_{13};\ S_{15}$.

9. $a_1 = -8, d = 6, a_n = 112;\ n;\ S_n$.

10. $a_1 = 17, S_{18} = 2601;\ d;\ a_{18}$.

11. $d = -7, S_{25} = 400;\ a_1;\ a_{25}$.

12. $a_1 = 27, a_n = 48, S_n = 1500;\ n;\ d$.

13. $a_1 = 15, d = -\frac{3}{2}, S_n = 28\frac{1}{2};\ n;\ a_n$.

14. $d = -4\frac{1}{2}, S_{17} = 136;\ a_1;\ a_{17}$.

15. $a_1 = 20, S_{20} = 20;\ d;\ a_{20}$.

16. $a_{29} = 10, d = \frac{3}{4};\ a_1;\ S_{29}$.

17. Find the arithmetic mean of $k^2 - 7km + m^2$ and $k^2 + 7km - 3m^2$.

18. Find the arithmetic mean of $x^2 + 4ab - y^2$ and $y^2 + 2ab - x^2$.

19. The eleventh and twenty-ninth terms of an arithmetic progression are 54 and 180, respectively. Find the forty-second term.

20. The twenty-seventh and sixty-first terms of an arithmetic progression are 238 and 408, respectively. Find the forty-first term.

21. The thirteenth and twenty-third terms of an arithmetic progression are 60 and 75, respectively. Find the thirty-seventh term.

22. The seventh and fifty-first terms of an arithmetic progression are 15 and -18, respectively. Find the twenth-third term.

23. Insert five arithmetic means between 7 and 79.

24. Insert six arithmetic means between -4 and 27.

25. Insert four arithmetic means between $-10 - 24\sqrt{3}$ and $-4\sqrt{3}$.

26. Insert three arithmetic means between $a - 4b$ and $7a - 2b$.

27. How many numbers between 15 and 600 are exactly divisible by 13? Find the sum of these numbers.

28. How many numbers between 6 and 500 are exactly divisible by 11? Find the sum of these numbers.

29. A harmonic progression is a sequence of numbers whose reciprocals form an arithmetic progression. Insert three harmonic means between 5 and 9.

30. If x^2, y^2, and z^2 are in arithmetic progression (note Problem 29), show that $x + y$, $z + x$, and $y + z$ form a harmonic progression.

31. A club raffles a gun by selling one hundred sealed tickets numbered in order 1, 2, 3, $\cdots$. These tickets are drawn at random by purchasers who pay the number of cents equal to the number on the ticket. How much money does the club receive?

32. It is estimated that a certain property, now valued at \$40,000, will depreciate as follows: \$1450 the first year, \$1400 the second year, \$1350 the third year, and so on. Based on these estimates, what will be the worth of the property fifteen years from now?

33. An interesting type of sale is often conducted at a certain camera shop. The price of an article is decreased 1 cent the first day, 2 additional cents the second day, 3 additional cents the third day, and so on until either the article is sold or the sale ends. Until what day of the sale must a customer wait to purchase an article originally priced at \$9.31, if he is willing to pay \$7.00 for it?

34. On a certain construction project a contractor was penalized for taking more than the contractual time to finish the project. He forfeited \$75 the first day, \$90 the second day, \$105 the third day, and so on. How many additional days did he need to complete the project if he paid a penalty of \$1215?

35. A man is offered a position at a salary which starts at \$4600 a year and increases yearly by \$250. How much would his total earnings amount to if he worked 12 years under this salary schedule?

36. The rungs of a ladder decrease uniformly in length from 2 ft 8 in. to 18 in. What is the total length of the wood in the rungs if there are twenty-five of them?

37. A man invests \$500 at the end of each year for fourteen years. How much money will he have at the end of that time if his money draws 2 per cent simple interest?

38. The sum of three numbers in arithmetic progression is 36. Find the numbers if the sum of their squares is 482. (HINT: Use $x - d$, x, $x + d$ for the numbers.)

39. Prove that the sum of the first n positive integers divisible by 6 is $3n(n + 1)$.

40. A body starting from rest falls 16.1 ft in the first second, 48.3 ft in the next second, 80.5 ft in the third second, and so on. How far does it fall in the eleventh second? How far does it fall in 22 seconds?

41. A rack in the form of an equilateral triangle is filled with balls so that there are eleven on a side. Another layer of balls is placed on this layer so that there are ten on a side; a third layer of balls with nine on a side is then placed on the second layer; and so on. Thus a pyramid of balls is formed, with one ball on top. How many balls are there in the bottom layer? How many balls are there in the pyramid?

87. Geometric Progressions

A **geometric progression** is a sequence in which the ratio of any term to the preceding term is a fixed number called the **common ratio.**

Illustrations: (*a*) The sequence 2, 4, 8, 16, 32 is a geometric progression whose common ratio is 2.

(*b*) The sequence 1, -0.1, 0.01, -0.001, 0.0001 is a geometric progression whose common ratio is -0.1.

A geometric progression may be completely described by specifying the following numbers:

a_1, the first term,

r, the common ratio,

and n, the number of terms.

With the use of these numbers, the first n terms may be written:

$$a_1,\ a_1r,\ a_1r^2,\ a_1r^3,\ \cdots,\ a_1r^{n-1}.$$

Besides the three numbers above,

a_n, the nth term,

and S_n, the sum of n terms,

are of interest here as in the arithmetic progressions. An important relation among four of the five **elements** a_1 a_n, r, n, and S_n of a geometric progression is

$$a_n = a_1r^{n-1}. \tag{3}$$

This formula is obtained directly from the definition; for the second term is obtained from the first by multiplication by r, the third from the first by multiplication by r^2, and so on, the number of multiplications of a_1 by r always being the number of the term diminished by 1.

The sum of n terms is given by the equation

$$S_n = a_1 + a_1r + a_1r^2 + \cdots + a_1r^{n-2} + a_1r^{n-1}.$$

If we multiply both sides of this equation by r, we get

$$rS_n = a_1r + a_1r^2 + a_1r^3 + \cdots + a_1r^{n-1} + a_1r^n.$$

The subtraction of the members of the second equation from those of the first yields

$$S_n - rS_n = a_1 - a_1r^n,$$

or

$$S_n(1 - r) = a_1(1 - r^n).$$

Now, if $r \neq 1$, we have

$$S_n = \frac{a_1(1 - r^n)}{1 - r}. \tag{4}$$

(If $r = 1$, the progression will consist of n terms, each equal to a_1. In this case, $S_n = na_1$.)

EXAMPLE 1. Find the tenth term and the sum of the first ten terms of the geometric progression whose first term is 1 and whose common ratio is 2.

Solution: Using (3), we have

$$a_{10} = a_1r^{10-1} = (1)(2^9) = 512. \quad \textit{Ans.}$$

Also, from (4) we have

$$S_{10} = \frac{a_1(1 - r^n)}{1 - r} = \frac{1(1 - 2^{10})}{1 - 2} = \frac{-1023}{-1} = 1023. \quad \textit{Ans.}$$

EXAMPLE 2. The sum of the first five terms of a geometric progression is $1\frac{3}{8}$, and the common ratio is $-\frac{1}{2}$. Find the terms.

Solution: From (4), we have

$$\frac{11}{8} = \frac{a_1[1 - (-\frac{1}{2})^5]}{1 - (-\frac{1}{2})},$$

or

$$\frac{11}{8} = \frac{11}{16}a_1.$$

Hence, $a_1 = 2$, and the progression is $2, -1, \frac{1}{2}, -\frac{1}{4}, \frac{1}{8}$. *Ans.*

The terms which occur between a_1 and a_n in a geometric progression are called **geometric means.** The student may show that a single geometric mean between two numbers is the same as a mean proportional between the two numbers.

EXAMPLE 3. Insert three geometric means between 16 and 81.

Solution: Counting the two given terms, we must have a progression of five terms, where $a_1 = 16$, and $a_5 = 81$.

Hence,

$$81 = 16r^4,$$

or

$$r^4 = \frac{81}{16},$$

and $$r = \frac{3}{2} \quad \text{or} \quad r = -\frac{3}{2}.$$

Thus, the required means are 24, 36, 54 or -24, 36, -54. *Ans.*

EXERCISES 72

1. Determine which of the following sequences are geometric progressions and give the value of the common ratio for each such progression:

(*a*) 14, 42, 126
(*b*) 216, 36, 6
(*c*) 320, -80, 20
(*d*) 147, -21, 3
(*e*) 18, 12, 8
(*f*) $\frac{2}{5}, \frac{4}{25}, \frac{8}{125}$
(*g*) $\frac{4}{9}, \frac{1}{6}, \frac{1}{16}$
(*h*) $\frac{2}{3}, \frac{2}{15}, \frac{2}{45}$

2. Find the eighth and twelfth terms and the sum of the first twelve terms of the geometric progression 6, 12, 24, 48, $\cdots$.

3. Find the sixth and eighth terms and the sum of the first eight terms of the geometric progression 2, 6, 18, $\cdots$.

4. Find the eighth term of the geometric progression

$$\sqrt{3},\ \sqrt{6},\ 2\sqrt{3},\ 2\sqrt{6},\ \cdots.$$

5. Find the seventh term of the geometric progression

$$\frac{a^{14}}{b^{20}},\ \frac{a^{12}}{3b^{17}},\ \frac{a^{10}}{9b^{14}},\ \cdots.$$

In each of Examples 6 to 14 certain elements of a geometric progression are given. Find the indicated elements whose values are not given.

6. $a_1 = 16,\ r = \frac{1}{2},\ n = 7;\ a_7;\ S_7.$

7. $a_1 = 5,\ S_4 = 200;\ a_4;\ r.$

8. $a_1 = 32,\ r = \frac{3}{2},\ S_n = 422;\ n;\ a_n.$

9. $r = -2,\ S_9 = -513;\ a_1;\ a_9.$

10. $a_7 = \frac{64}{9},\ r = \frac{2}{3};\ a_1;\ S_7.$

11. $a_8 = -\frac{1}{16},\ r = -\frac{1}{4};\ a_1;\ S_8.$

12. $a_1 = \frac{243}{256},\ a_n = 3,\ S_n = 9\frac{39}{256};\ r;\ n.$

13. $a_1 = \frac{7}{4},\ a_n = 112,\ S_n = 222\frac{1}{4};\ r;\ n.$

14. $a_1 = 250,\ r = \frac{3}{5},\ a_n = 32\frac{2}{5};\ n;\ S_n.$

15. Write the first three terms of a geometric progression in which the fourth term is 2 and the seventh term is 54.

16. Write the first four terms of a geometric progression in which the fifth term is $\frac{1}{7}$ and the seventh term is $\frac{4}{343}$.

17. What are the single geometric means between $\frac{1}{9}$ and $3\frac{6}{25}$?

18. Find single geometric means between a^2 and b^4.

19. Insert three geometric means between 2 and 18.

20. Insert four geometric means between $10\frac{2}{3}$ and $\frac{1}{3}$.

21. Insert five geometric means between 7 and 56.

22. Show that Formula (4) for S_n may be written $S_n = \dfrac{a_1 - ra_n}{1 - r}$.

23. If the population of a certain country increases at the rate of 5 per cent per year and the present population is 300,000, what will the population be 6 yr from now?

24. The number of bacteria in a culture increased from 320,000 to 2,430,000 in 5 days. Find the daily rate of increase if this rate is assumed to be constant.

25. A man invested a certain sum of money which in the second year earned $1\frac{1}{4}$ times as much as in the first year, and in the third year earned $1\frac{1}{4}$ times as much as in the second year, and so on. If the investment earned \$9225 in the first 4 yr, how much did it yield in the second and fourth years?

26. A savings bank pays interest at the rate of 2 per cent compounded annually. What will be the total accumulation at the end of 7 yr if \$800 is deposited in the bank now?

27. The distance traveled by a point on a compound pendulum in any swing is 20 per cent less than that in the preceding swing. If the length of the first swing is s, what fractional part of s is the arc covered in the seventh swing? Find the total distance covered in seven swings in terms of s.

28. If three numbers in arithmetic progression are increased by 1, 4, and 43, respectively, the resulting numbers are in geometric progression. Find the original numbers if their sum is 36.

29. If three numbers in arithmetic progression are increased by 9, 7, and 9, respectively, the resulting numbers are in geometric progression. Find the original numbers if their sum is 3.

30. In a certain chemical plant, a tank holds 100 gal of a liquid that mixes readily with water. After 25 gal of the liquid are drawn out, the tank is filled by replacing the liquid with water. Then 25 gal of the resulting mixture are drawn out, and the tank is again filled with water. If the operation is performed until six batches have been drawn from the tank, how much of the original liquid remains?

31. Each of a series of equilateral triangles has its vertices at the midpoints of the sides of the preceding triangle. Find the length of the side and the area of the eighth triangle if a side of the largest triangle is 32 in. long.

32. A square is inscribed in a circle of diameter 64 in. Then another circle is inscribed in this square, and a second square is inscribed in this circle, and so on. Find the area of the seventh square.

33. An equilateral triangle is circumscribed about a circle with a radius of $\frac{1}{2}$ in. after which another circle is circumscribed about the equilateral triangle. A second equilateral triangle is circumscribed about the second circle, and so on. Find the area of the seventh circle.

88. Infinite Geometric Series

Formula (4) of the preceding section gives the sum of the first n terms of a geometric progression as

$$S_n = \frac{a_1(1 - r^n)}{1 - r} = a_1 + a_1r + a_1r^2 + \cdots + a_1r^{n-1}, \text{ if } r \neq 1.$$

We now consider what happens if n is allowed to increase without

bound. We indicate that the number of terms is without limit by writing

$$a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-1} + \cdots.$$

This collection of symbols is known as an **infinite geometric series.** The word "infinite" means only that the terms of the series proceed without end.

In the formula for S_n, the number n occurs only in r^n. Now, if r is numerically less than 1, r^n will become smaller and smaller in absolute value as n is taken increasingly large. For example, if $r = 0.6$, we have, approximately,

$$r^{10} = (0.6)^{10} = 0.00605;$$

$$r^{100} = (0.6)^{100} = (6.53)(10^{-23});$$

and

$$r^{1000} = (0.6)^{1000} = (1.4)(10^{-222}).$$

(In the last two values, there are 22 and 221 zeros, respectively, between the decimal point and the first significant digit.)

It therefore appears that when $|r| < 1$, S_n may be made to become and remain as close to $\dfrac{a_1}{1-r}$ as we wish by taking n large enough. In this case, we write

$$\lim_{n\to\infty} S_n = \frac{a_1}{1-r}, \ |r| < 1. \qquad (5)$$

The symbol $\lim\limits_{n\to\infty} S_n$ is read "the limit of S_n as n increases without limit."

In case $|r| > 1$, r^n increases indefinitely in absolute value as n increases, and, therefore, the value of S_n cannot be made to become and remain arbitrarily close to any fixed number. If $r = 1$, $S_n = na_1$ again increases indefinitely in absolute value if n increases without limit. Finally, if $r = -1$, $S_n = a_1$ or 0, according as n is odd or even, and does not become and *remain* arbitrarily close to any number as n increases without limit. The geometric series is of no use to us under any of these latter circumstances.

A simple geometrical interpretation may be made of the behavior of S_n, as n increases without limit, if $r = \frac{1}{2}$. We take $a_1 = 1$ for simplicity and consider the series

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \left(\frac{1}{2}\right)^{n-1} + \cdots.$$

For this series, we have

$$S_1 = 1,$$

$$S_2 = 1 + \frac{1}{2},$$

$$S_3 = 1 + \frac{1}{2} + \frac{1}{4},$$

$$S_4 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8},$$

and so on.

In Figure 42, the points corresponding to the S's have been marked. We see that each step cuts in half the remaining distance to the point

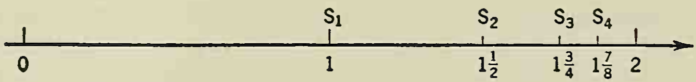


Fig. 42

marked 2. It is clear that by taking n sufficiently large and keeping it so, the value of S_n may be made to become and remain as close to 2 as we wish. For this series, formula (5) gives

$$\lim_{n\to\infty} S_n = \frac{a_1}{1 - r} = \frac{1}{1 - \frac{1}{2}} = 2.$$

We frequently abbreviate $\lim\limits_{n\to\infty} S_n$ by S and call it the "sum" of the geometric series. Notice, however, that S is a sum only in an extended sense of the word; it cannot be attained by taking any definite number of terms.

Example 1. Find the sum of the geometric series

$$2 - \tfrac{4}{3} + \tfrac{8}{9} - \cdots.$$

Solution: For the given series,

$$a_1 = 2 \quad \text{and} \quad r = -\tfrac{2}{3}.$$

Hence, $$S = \frac{2}{1 - (-\frac{2}{3})} = \frac{2}{\frac{5}{3}} = \frac{6}{5}\cdot \quad Ans.$$

By a *repeating decimal,* we mean an endless decimal which from some

point on consists of a repeated sequence of digits. For example, $0.3333 \cdots$ and $1.6\ 23\ 23\ 23 \cdots$ are repeating decimals.

Example 2. Find the common fraction whose decimal representation is $1.6\ 23\ 23\ 23 \cdots$.

Solution: $1.6\ 23\ 23\ 23 \cdots = 1.6 + 0.023 + 0.00023 + 0.000\ 0023 + \cdots$

Beginning with 0.023, we have an infinite geometric series whose first term is 0.023 and whose ratio is 0.01. Hence,

$$1.6\ 23\ 23\ 23 \cdots = 1.6 + \frac{0.023}{1 - 0.01}$$

$$= \frac{16}{10} + \frac{23}{990} = \frac{1607}{990}\cdot \quad \textit{Ans.}$$

EXERCISES 73

Find the sum of each of the following infinite geometric series:

1. $\frac{7}{10} + \frac{7}{100} + \frac{7}{1000} + \frac{7}{10{,}000}$

2. $\frac{17}{10^2} + \frac{17}{10^4} + \frac{17}{10^6} + \cdots$

3. $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$

4. $1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{64} + \cdots$

5. $25 - 5 + 1 - \frac{1}{5} + \cdots$

6. $64 + 24 + 9 + 3\frac{3}{8} + \cdots$

Find the common fraction equivalent to each of the given repeating decimals in Exercises 7 to 16. Verify your answer by division.

7. $0.444444 \cdots$

8. $1.555555 \cdots$

9. $0.67\ 67\ 67 \cdots$

10. $0.53\ 53\ 53 \cdots$

11. $3.48\ 48\ 48 \cdots$

12. $5.08\ 08\ 08 \cdots$

13. $7.1\ 27\ 27\ 27 \cdots$

14. $0.02\ 54\ 54\ 54 \cdots$

15. $0.279\ 279\ 279 \cdots$

16. $4.729\ 729\ 729 \cdots$

17. A rubber ball is dropped from a height of 8 ft. If the ball rebounds each time to a height equal to three fourths of the preceding height, what total distance will the ball travel before coming to rest?

18. A pendulum on each separate swing describes an arc whose length is 0.98 of the length of the preceding arc. If the length of the first arc is 12 in., what distance is traversed by the pendulum before coming to rest?

89. Investment Problems

Many of the important problems of business and investment can be solved by means of arithmetic and geometric progressions. For pur-

poses of reference, we repeat the simple-interest and the compound-interest formulas.

$$\text{Simple interest: } A = P(1 + ni).$$

This gives the accumulation A for a principal P at the simple interest rate i for n years.

$$\text{Compound interest: } A = P(1 + i)^n.$$

This gives the accumulation A for a principal P at the interest rate i compounded annually for n years.

EXAMPLE 1. A man deposits \$1000 in a bank at the beginning of each year. Find his total savings immediately after his tenth deposit (*a*) at 2 per cent, simple interest; (*b*) at 2 per cent interest, compounded annually.

Solution: (*a*) The amount deposited at the beginning of the tenth year would earn no interest; the amount deposited at the beginning of the ninth year would earn interest for 1 yr; that at the beginning of the eighth year would earn interest for 2 yr, and so on. Thus, the total accumulation may be written

$$S_{10} = R + R(1 + i) + R(1 + 2i) + \cdots + R(1 + 9i),$$

where $R = 1000$ and $i = 0.02$.

The expression for the value of S_{10} is the sum of an arithmetic progression with $a_1 = R$ and $a_{10} = R(1 + 9i)$. Hence,

$$\begin{aligned} S_{10} &= \tfrac{10}{2}[R + R(1 + 9i)] = 5R(2 + 9i) \\ &= (5000)(2.18) = \$10{,}900. \quad \textit{Ans.} \end{aligned}$$

(*b*) The argument in this part is the same as in (*a*) with the exception that the compound-interest formula is used. Thus,

$$S_{10} = R + R(1 + i)^1 + R(1 + i)^2 + \cdots + R(1 + i)^9.$$

In this case, the progression is geometric with $1 + i$ as the common ratio. Hence,

$$\begin{aligned} S_{10} &= \frac{R[1 - (1 + i)^{10}]}{1 - (1 + i)} \\ &= \frac{R[(1 + i)^{10} - 1]}{i} \\ &= \frac{1000[(1.02)^{10} - 1]}{0.02} = \$10{,}949.72. \quad \textit{Ans.} \end{aligned}$$

Note: In problems such as part (*b*) of the preceding example, tables are commonly used to obtain answers correct to the nearest cent. For present purposes, the student may obtain an approximate answer either by the use of the binomial formula or by employing logarithms for the computation of $(1 + i)^n$.

Investment problems frequently involve interest calculations more than once each year. In this connection, the phrase "k per cent compounded m times per year" means that the actual interest rate is $\frac{k}{m}$ per cent per interest period. For example, the phrase "4 per cent, compounded semiannually" means that 2 per cent interest is earned each half year, at the end of which time the accrued interest is made part of the principal; the total amount then earns 2 per cent interest for the next half year, and so on. In any such problem, we may use the same compound-interest formula as before if we put $i = \frac{k}{m}$ per cent and let n be the total number of interest periods.

Example 2. A fund of \$20,000 is to be used to make five equal payments, one at the end of each 6 mo starting from the present time. If the fund earns interest at the rate of 8 per cent, compounded semiannually, find the value of each of the equal payments.

Solution: Imagine the \$20,000 divided into five parts P_1, P_2, P_3, P_4, and P_5; then, P_1 is to be allowed to accumulate for 6 mo to make the first payment, say R; P_2 is to accumulate for 1 yr to make the second payment, also R; and so on. This means that

$$R = P_1(1 + i) \quad \text{or} \quad P_1 = R(1 + i)^{-1},$$

$$R = P_2(1 + i)^2 \quad \text{or} \quad P_2 = R(1 + i)^{-2},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$R = P_5(1 + i)^5 \quad \text{or} \quad P_5 = R(1 + i)^{-5}.$$

Since $P_1 + P_2 + \cdots + P_5 = 20{,}000$, we have

$$R[(1 + i)^{-1} + (1 + i)^{-2} + \cdots + (1 + i)^{-5}] = 20{,}000.$$

The quantity in brackets is the sum of a geometric progression with $a_1 = (1 + i)^{-1}$ and $r = (1 + i)^{-1}$. Therefore,

$$R(1 + i)^{-1}\left[\frac{1 - (1 + i)^{-5}}{1 - (1 + i)^{-1}}\right] = 20{,}000,$$

or
$$R\left[\frac{1-(1+i)^{-5}}{i}\right] = 20{,}000,$$

and
$$R = \frac{20{,}000i}{1-(1+i)^{-5}}.$$

In this problem, the rate is 8 per cent, compounded semiannually;

hence,
$$i = \frac{0.08}{2} = 0.04,$$

and
$$R = \frac{(20{,}000)(0.04)}{1-(1.04)^{-5}} = \frac{800}{1-0.82196}$$
$$= \frac{800}{0.17804} = \$4493. \quad \textit{Ans.}$$

This answer was obtained by five-place logarithms. Investment tables give \$4492.54.

If the compound-interest formula is solved for P, we have

$$P = A(1+i)^{-n}.$$

The factor $(1+i)^{-n}$ is called the *compound-discount factor*. The preceding formula gives the *present value* of an amount A at compound discount, that is, the principal P that would have to be deposited now in order to accumulate to the amount A in n periods at the stated rate.

Example 3. A man offers to sell his house for (a) \$10,000 cash or (b) three payments of \$3500 each, one in cash, one at the end of 1 yr and one at the end of 2 yr. What rate of interest, compounded annually, is the man asking in plan (b)?

Solution: If we subtract the cash payment of \$3500 from the total cash price, we have \$6500 which may be considered as the sum of the present values of the remaining payments. Hence,

$$6500 = 3500(1+i)^{-1} + 3500(1+i)^{-2}.$$

We multiply both sides of this equation by $(1+i)^2$ and simplify to obtain

$$13(1+i)^2 - 7(1+i) - 7 = 0,$$

a quadratic equation in $(1 + i)$. From this equation we find

$$1 + i = \frac{7 \pm \sqrt{49 + 364}}{26}$$

$$= \frac{7 \pm 20.322}{26}.$$

Since $1 + i$ must be a positive number, we discard the minus sign and have

$$1 + i = \frac{27.322}{26} = 1.051-.$$

Therefore, $$i = 0.051-,$$

and the man is asking approximately 5.1 per cent, compounded annually. *Ans.*

EXERCISES 74

Where two answers are given for any of the following problems, the first is computed by means of five-place logarithms and the second by means of investment tables.

1. Find the accumulated amount just after the last payment into a fund built up by deposits of \$250 at the end of each year for 12 yr if the interest rate is 3 per cent, compounded annually.

2. A fund is accumulated by making a deposit of \$100 at the end of each 6 mo for $7\frac{1}{2}$ yr. What amount is in the fund just after the last payment if the interest rate is 2 per cent compounded semiannually?

3. A man places \$500 at the end of each 3 mo in a bank that pays interest at the rate of 2 per cent, compounded quarterly. How much will the bank owe him just after his twenty-fourth deposit?

4. To what amount will an investment of \$500 at the end of each quarter for $5\frac{1}{2}$ yr accumulate if money is worth 4 per cent, compounded quarterly?

5. Find the present value of the annuity which pays \$2000 at the end of each year for 10 yr, starting now, if the interest rate is 2 per cent, compounded annually.

6. Find the present value of the annuity which pays \$600 at the end of each 6 mo for $7\frac{1}{2}$ yr, if the interest rate is $2\frac{1}{2}$ per cent compounded semiannually.

7. What sum of money should a person invest at 5 per cent, compounded annually, to yield \$2800 a year at the end of each year for 9 yr, the first payment being 1 yr from now?

8. How much money should a person invest at 4 per cent, compounded semiannually, to yield \$1200 at the end of every 6 mo for 12 yr, the first payment being 1 yr from now?

9. What semiannual pension for 15 yr will \$9000 buy if money is worth 4 per cent, compounded semiannually?

10. What annual pension for 20 yr will \$25,000 buy if money is worth 3 per cent, compounded annually?

11. A person buys a farm and agrees to pay \$15,000 down and \$2000 at the end of each year for 8 yr. What is the cash value of the farm if money is worth 4 per cent, compounded annually? compounded semiannually?

12. To create a fund of \$200,000 at the end of 15 yr, what equal payments must a corporation deposit at the end of each year if money is worth 3 per cent, compounded annually?

13. A property is sold for a down payment of \$7000, with \$700 to be paid at the end of each year for 20 yr. What is the present value of the property if the interest on the payments is 4 per cent, compounded annually?

14. If a man now has a debt of \$8000, how much must he pay at the end of each year to discharge the debt in ten equal payments, the interest rate being 5 per cent, compounded annually?

15. A university builds a dormitory and plans to discharge a debt of \$1,000,000 in 25 yr by twenty-five equal payments to be made at the end of each year. If money is worth 3 per cent, what is the amount of each payment?

16. A person purchased a property 8 yr ago for \$5200 and 4 yr ago made \$2000 worth of improvements. At the end of each year he paid \$130 in taxes. What is his present investment in the property if the interest is figured at 3 per cent, compounded annually?

17. A manufacturer buys \$100,000 worth of new machinery. He makes a cash payment of \$50,000 and contracts to pay \$27,500 at the end of each year for the next 2 yr. What rate of interest, compounded annually, is being charged?

18. A house can be bought for \$20,000 cash or for an \$8000 down payment and \$6480 paid at the end of each year for the next 2 yr. Find what rate of interest, compounded annually, is being charged?

19. A person invests \$1000 at the end of each year for 5 successive years. This total amount is then returned to him in equal payments at the end of each year during the next 10 years. What is the amount of each payment if money is worth 5 per cent, compounded annually?

20. A person makes equal payments at the end of each year for 10 yr, for which he receives an annuity of \$400 at the end of each year for the following 20 yr. If money is worth 4 per cent, compounded annually, what is the amount of each of the equal payments made during the first 10 yr?

Chapter 15

COMPLEX NUMBERS

90. Introduction

We have previously met the concept of a complex number in connection with the operation of extracting square roots. In order to make this operation always possible for real numbers, we defined the two square roots of -1 as i and $-i$ so that $i^2 = -1$.

Any number of the form $a + bi$, where a and b are real, is called a **complex number.** The *real part* of $a + bi$ is a, and the *imaginary part* is bi; b is the coefficient of the imaginary part. If $b = 0$, the number is an ordinary real number, and if $a = 0$ and $b \neq 0$, the number is called a **pure imaginary** number.

Complex numbers may be combined according to the usual laws of the fundamental operations with the additional rule that wherever i^2 occurs, it is replaced by -1. As we have seen, the combination of two complex numbers by addition, subtraction, multiplication, or division leads to another complex number. Since the real numbers constitute a special class of complex numbers, it appears that the complex numbers give us a true extension of the number system of algebra. It may be further noted that the system of complex numbers is the most general system of numbers to which all the ordinary rules of algebra apply. In fact, it has been shown that every more general number system fails to obey at least one of the laws of the fundamental operations.

Thus, we may think of ordinary algebra as being that algebra which is based on the complex numbers; no further extension of the number system is needed to handle the problems of this basic algebra. For example, not only has the introduction of complex numbers made it

possible for us to extract the square roots of every real number, but it has also given us a more complete picture of the behavior of the quadratic equation in one unknown. Had it not been for the use of complex numbers, we should have had to say that some quadratic equations, such as $x^2 + 1 = 0$, have no roots, whereas others, such as $x^2 - 1 = 0$, have roots.

The discussion in this chapter is intended to give a clearer picture of the meaning and importance of complex numbers, on the one hand, by means of their interpretation as vectors, and, on the other hand, by a consideration of the geometric meaning of the operation of multiplication of one complex number by another.

91. Vectors in a Plane

With reference to Figure 43, suppose that a point originally at the origin is displaced 3 units in the positive X direction and then 4 units in the positive Y direction. The net effect is a displacement of 5 units in the direction which makes the angle θ with the positive X axis as shown in the diagram. This displacement is characterized completely by giving either the distance and direction of the final displacement or the two displacements first described.

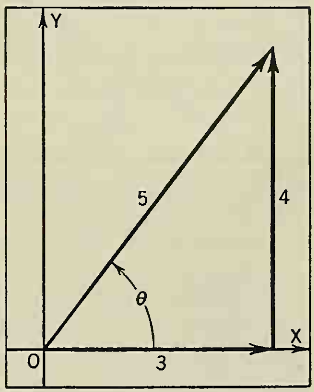


Fig. 43

Any quantity which requires both **magnitude** and **direction** for its description is called a **vector quantity**. Examples of such quantities are displacements, forces, velocities, and accelerations. A vector quantity may be represented by an arrow whose length is proportional to the magnitude, and whose direction is the direction of the vector quantity. An arrow used for this purpose is called a **vector**. We shall be concerned with vectors pointing out from the origin in a plane which we shall call the XY plane.

Generalizing a statement made above, we see that a vector is completely characterized by giving its length and the angle it makes with the positive X axis. The angle may be regarded as generated by rotating a line in the counterclockwise sense about the origin from a

position of coincidence with the positive X axis to a position of coincidence with the line of the arrow.

From the arrow end of any vector, such as OP in Figure 44, perpendiculars may be dropped to the two axes. The directed distances from the origin to the feet of these perpendiculars are called **components** of the vector. In Figure 44, OA is the *X component,* and OB is the *Y component* of the vector OP. The sign attached to OA is that of the x coordinate of the point P, and the sign attached to OB is that of the y coordinate of P. Since OA and OB have both magnitude and direction, they are also vectors. If OA and OB are given, the coordinates of the arrow end of OP are given; hence, OP is determined. In this sense, the two vectors OA and OB together are equivalent to the vector OP.

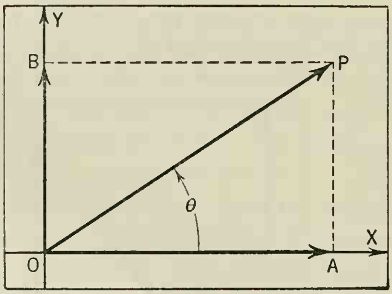


Fig. 44

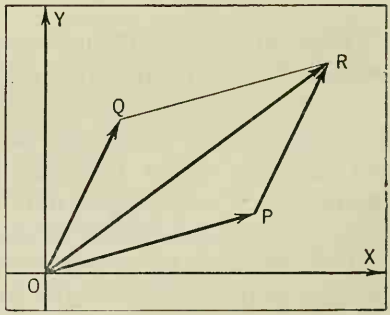


Fig. 45

A displacement from O to P followed by a displacement from P to R (see Figure 45) is equivalent to a displacement from O to R. The vector OR is called the **resultant** of the vectors OP and PR. If we draw a vector OQ parallel to PR and equal to PR in length, the line QR will complete a parallelogram of which OR is a diagonal. The **parallelogram law** for combining vectors states that *the resultant of two vectors is given in magnitude and direction by the diagonal of the parallelogram just described.* The process of finding the resultant of two vectors is called **vector addition.**

EXAMPLE 1. Describe the resultant of the following two forces acting upon a point located at the origin: a 5-lb force in the positive X direction and a 12-lb force in the negative Y direction.

Solution: By the parallelogram law, the resultant is given by the

diagonal OP in Figure 46. The magnitude of the resultant is

$$\sqrt{5^2 + 12^2} = \sqrt{169} = 13 \text{ lb.} \quad Ans.$$

The student will learn in trigonometry how to calculate the angle θ, or, if the figure is drawn carefully to scale, its approximate measure may be obtained by the use of a protractor. For present purposes, it will be sufficient to indicate the angle in the diagram.

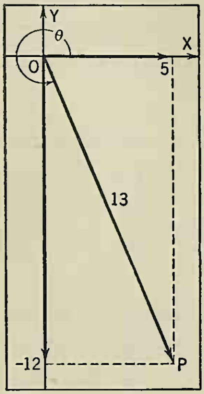


Fig. 46

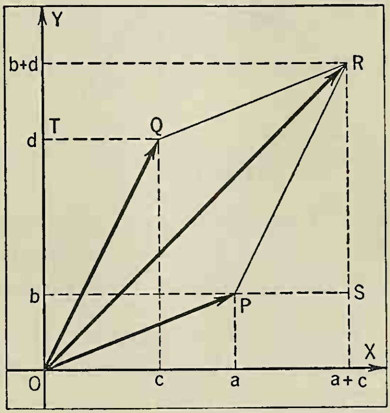


Fig. 47

From the previous discussion, it follows that two vectors not in the same line must always be added by means of the parallelogram law. However, vectors in the same straight line are added in the manner of signed numbers on the number scale, as may be seen from the following examples. Two forces in the positive X direction, one of 5 lb and one of 7 lb, are equivalent to a single force of 12 lb in the same direction. A force of 5 lb in the positive X direction and a force of 7 lb in the negative X direction are equivalent to a force of 2 lb in the negative X direction. Similar statements may be made concerning two forces in any straight line.

Let us consider two forces acting on the same point, located at the origin: one with an X component a and a Y component b, and a second with an X component c and a Y component d. The vectors OP and OQ in Figure 47 represent these two forces; and the vector OR represents the force whose components are $a + c$ and $b +$ d, respectively. We may show that OR is the resultant of OP and OQ as follows: Horizontal lines are drawn through Q and P and are labeled TQ and PS, respectively. A vertical line is drawn through R, meeting PS at S. Since d is the

length of both OT and SR and c is the length of both TQ and PS, triangles OTQ and PSR are congruent right triangles. Hence, OQ is equal in length to PR. Furthermore, angle RPS = angle TQO, so that OQ is parallel to PR. Therefore, $POQR$ is a parallelogram, and OR must be the required resultant.

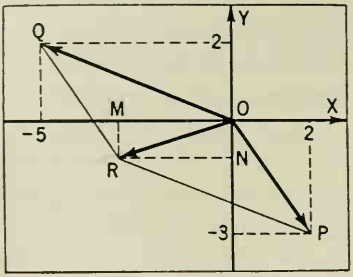


Fig. 48

The preceding paragraph may be summarized by the following statement. *The* X *and* Y *components of the resultant of two vectors are the sums of the corresponding components of the vectors.*

EXAMPLE 2. A vector has an X component of 2 and a Y component of -3. A second vector has corresponding components -5 and $+2$, respectively. Find the magnitude of the resultant of the two vectors if both are acting upon a point located at the origin.

Solution: The diagram is shown in Figure 48, from which we may read that the X component of the resultant is $OM = +2 - 5 = -3$ and the Y component is $ON = -3 + 2 = -1$. So the magnitude of the resultant OR is

$$\sqrt{(-3)^2 + (-1)^2} = \sqrt{10}. \quad \textit{Ans.}$$

EXERCISES 75

1. Find the magnitude of the resultant of a force of 15 lb acting in the negative X direction and a force of 8 lb acting in the positive Y direction.

2. Find the magnitude of the resultant of a force of 9 lb acting in the negative X direction, a force of 7 lb acting in the negative Y direction, and a force of 2 lb acting in the positive X direction.

3. Find the magnitude of the resultant of four forces acting at a point, and described as follows: a force of 8 lb downward, a force of 32 lb upward, a force of 13 lb to the left, and a force of 20 lb to the right.

4. A ship which is headed east strikes an ocean current flowing from north to south with a speed of 4 knots, that is, 4 nautical miles per hour. If the ship's speed in still water is 18 knots, what is the resultant speed? Show in a diagram the actual direction in which the ship is moving.

5. A jet plane is headed north at right angles to a west wind whose speed is 80 mph. If the plane's speed in the direction of the heading is 600 mph, what is the resultant speed of the plane? Use a diagram to show the actual direction in which the plane is traveling.

Find the magnitude of the resultant of each of the following sets of vectors. Make a diagram for each problem.

6. 14 units upward, 12 units to the right, and 5 units downward.

7. 10 units to the right, 8 units upward, and 4 units to the left.

8. 17 units to the right, 12 units upward, 11 units to the left, and 4 units downward.

9. 5 units to the right, 31 units upward, 12 units to the left, and 7 units downward.

10. 10 units upward and to the right at an angle of 45°, 4 units to the left and downward at an angle of 45°, and $3\sqrt{2}$ units to the left.

11. 8 units to the left, and $10\sqrt{2}$ units to the right and downward at an angle of 45°.

92. Interpretation of Complex Numbers as Vectors

A complex number in the form $a + bi$ may be represented graphically on a system of rectangular coordinates in the following manner: The real part a is plotted along the horizontal axis, and the imaginary coefficient b is plotted along the vertical axis. The point (a, b) thus determined may be taken as the graphical representation of the complex number $a + bi$. However, it is more convenient in an elementary course to regard the vector from the origin to the point (a, b) as a representation of the number see (Figure 49).

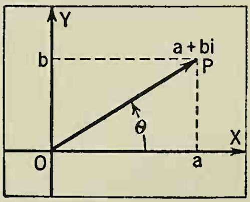


Fig. 49

With this second interpretation, it is at once evident that a and b may be thought of as the X and Y components, respectively, of a vector in the XY plane.

The magnitude of the vector OP, which is used to represent the number $a + bi$, is $\sqrt{a^2 + b^2}$. This magnitude, which is always a real, positive number, is called the **modulus,** or **absolute value,** of the complex number. To indicate the modulus, we shall use the symbol

$$|a + bi|$$

which may be read, the "modulus of $a + bi$." The angle θ between the vector and the positive X axis is referred to as the **angle of the complex number.**

93. Addition and Subtraction of Complex Numbers

It follows from the preceding sections that the addition of complex numbers corresponds to the addition of vectors in the plane, for the sum

of two complex numbers $a + bi$ and $c + di$ is $a + c + i(b + d)$, which may be regarded as the resultant of the two vectors representing $a + bi$ and $c + di$ in the manner shown in Section 91.

EXAMPLE 1. Add the complex numbers $2 + 3i$ and $-4 + 2i$. Draw a diagram showing the vector interpretation of the addition, and find the modulus of the sum.

Solution: The given complex numbers are combined by ordinary algebraic addition as follows:

$$\begin{array}{r} 2 + 3i \\ -4 + 2i \\ \hline -2 + 5i. \end{array} \quad \textit{Ans.}$$

The diagram is shown in Figure 50. The modulus of the sum is

$$|-2 + 5i| = \sqrt{(-2)^2 + 5^2} = \sqrt{29}. \quad \textit{Ans.}$$

In order to subtract a vector OP from a vector OQ, we reverse the direction of OP and add the result to OQ. The addition is accomplished

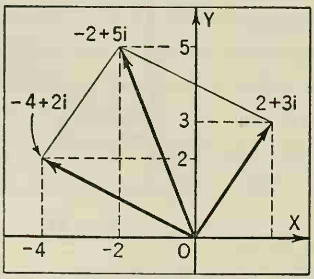


Fig. 50

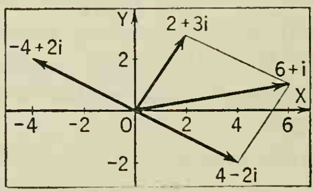


Fig. 51

by the parallelogram law. Hence, the subtraction of one complex number from another corresponds to the subtraction of one vector from another, as may be seen from the following example.

EXAMPLE 2. Subtract $-4 + 2i$ from $2 + 3i$. Illustrate by a diagram showing the vector interpretation.

Solution: $2 + 3i - (-4 + 2i) = 2 + 3i + 4 - 2i = 6 + i$. *Ans.* The diagram is shown in Figure 51.

EXERCISES 76

In each of the following exercises perform the indicated operations, draw a diagram showing the vector interpretation, and find the modulus of the answer:

1. $(2 + 3i) + (4 + 5i)$ **2.** $(1 + 5i) + (2 - i)$
3. $(4 - 2i) + (8 + 7i)$ **4.** $(-8 + 4i) + (6 - i)$
5. $(-10 - 4i) + (3 + 11i)$ **6.** $(-7 - 3i) + (3 + 6i)$
7. $(5 + 12i) - (3 + 4i)$ **8.** $(12 - 5i) - (4 + 3i)$
9. $(-4 - 5i) - (2 + i)$ **10.** $(-7 - i) - (-9 - 2i)$
11. $(10 - 2i) - (-14 - 9i)$ **12.** $(-8 + 4i) - (-7 + 5i)$
13. $(2 + 6i) + (5 - 2i) + (-2 - 4i)$
14. $(-4 + 3i) + (-3 - 7i) + (8 + 6i)$
15. $(9 + 2i) - (4 - 3i) + (-5 + 4i)$
16. $(-7 - 5i) - (-6 - 8i) - (-2 + 7i)$

94. Multiplication of Complex Numbers

There are certain applications, such as in electric-circuit theory, where it is desirable to think of the symbol bi, where b is real, as a product of b and the unit i. Figure 52 shows the number b and the number bi. From this figure it is clear that this "multiplication" by i turns the number b through 90° and leaves its magnitude unchanged.

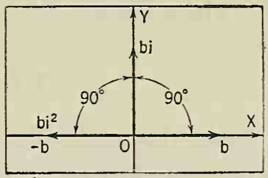


Fig. 52

A second multiplication by i may be indicated by bi^2. If we use the interpretation of a rotation through 90°, we see from Figure 52 that bi^2 and $-b$ are the same. Hence, our previous definition of

$$i^2 = -1$$

appears to be consistent with this new interpretation.

Following the idea of the preceding paragraph, we have

$$i^3 = i^2 \cdot i = -i;$$

$$i^4 = (i^2)^2 = (-1)^2 = +1;$$

$$i^5 = i(i^4) = i;$$

$$i^6 = i^2(i^4) = -1;$$

and so on.

By making use of the definition $i^2 = -1$, any positive or negative integral power of i may be reduced to one of the numbers i, -1, $-i$, or 1.

Example 1. Simplify (*a*) i^{103}; (*b*) i^{-103}.

Solution: (*a*) $i^{103} = (i^2)^{51} \cdot i = (-1)^{51} \cdot i = -1 \cdot i = -i$. *Ans.*

(*b*) $i^{-103} = i^{-104} \cdot i = (i^2)^{-52} \cdot i = (-1)^{-52} \cdot i = +1 \cdot i = i$. *Ans.*

Part (b) may also be handled as follows:

$$i^{-103} = \frac{1}{i^{103}} = \frac{1}{-i} \qquad \text{[See Part } (a)]$$

$$= \frac{1}{-i} \cdot \frac{i}{i} = \frac{i}{-(i^2)} = i.$$

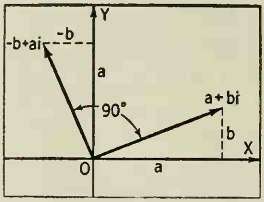


Fig. 53

We have seen previously that complex numbers are combined by the ordinary rules of algebra. Thus,

$$i(a + bi) = ai + bi^2 = -b + ai,$$

and Figure 53 makes it clear that the desired product is obtained by turning the number $a + bi$ through 90°.

For the multiplication of two complex numbers in general, we may write

$$(a + bi)(c + di) = a(c + di) + bi(c + di.)$$

Hence, the product may be found as the resultant of the two vectors $bi(c + di)$ and $a(c + di)$. This is shown in Figure 54b. Since a is a real number, it follows that the angle of the complex number $a(c + di)$ is the same as that of $c + di$, that is, the angle θ in Figure 54b is the same as the angle θ in Figure 54a. It may also be seen that the vector OQ makes an angle of 90° with the vector OR. Why?

Since the length of RP is the length of OQ, we have

$$RP = |bi(c + di)| = |bci - bd|$$

$$= \sqrt{(-bd)^2 + (bc)^2} = |b|\sqrt{c^2 + d^2}.$$

Similarly, $$OR = |a(c + di)| = |ac + adi|$$

$$= \sqrt{(ac)^2 + (ad)^2} = |a|\sqrt{c^2 + d^2}.$$

Referring to triangle OMA (Figure 54a) and triangle ORP (Figure 54b), we have

$$\frac{OR}{OM} = \frac{|a|\sqrt{c^2 + d^2}}{|a|} = \sqrt{c^2 + d^2},$$

and $$\frac{RP}{MA} = \frac{|b|\sqrt{c^2 + d^2}}{|b|} = \sqrt{c^2 + d^2}.$$

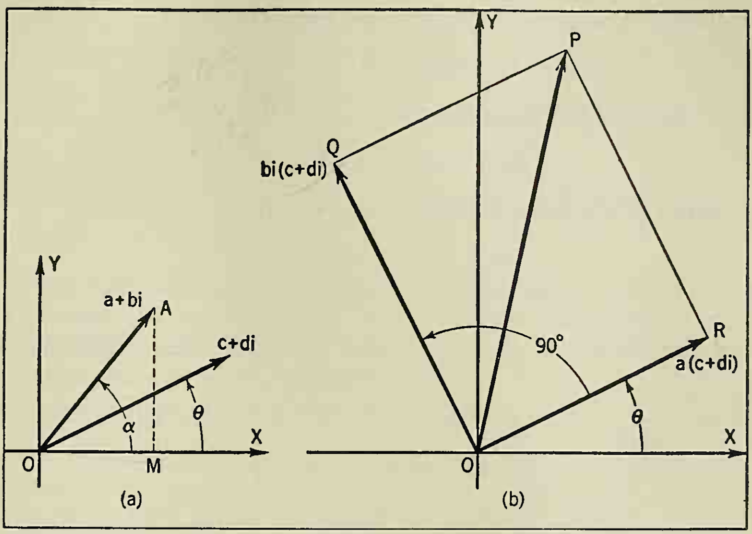


Fig. 54

Thus, the corresponding legs of triangles OMA and ORP are proportional, and the triangles are similar. This fact shows that angle ROP is equal to angle α; so OP makes the angle $\alpha + \theta$ with the X axis. Moreover,

$$\begin{aligned} OP &= |(a + bi)(c + di)| = |(ac - bd) + i(ad + bc)| \\ &= \sqrt{(ac - bd)^2 + (ad + bc)^2} \\ &= \sqrt{a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2} = \sqrt{a^2(c^2 + d^2) + b^2(c^2 + d^2)} \\ &= \sqrt{(a^2 + b^2)}\sqrt{(c^2 + d^2)}, \end{aligned}$$

which shows that the number of units in the length of OP is the product of the number of units in the length of OA and the number in the length of OB.

Although the angle α in Figure 54a has been taken as an acute angle, an appropriate figure leads to the corresponding result for any value of the angle. We therefore have the general theorem:

The modulus of the product of two complex numbers is the product of their moduli; the angle of the product is the sum of the angles of the factors.

Example 2. Verify the theorem above for the numbers $\sqrt{3} + i$ and $-1 + i\sqrt{3}$.

Solution: The product of the given numbers is

$$(\sqrt{3} + i)(-1 + i\sqrt{3}) = -2\sqrt{3} + 2i,$$

and the modulus of this product is

$$|-2\sqrt{3} + 2i| = \sqrt{12 + 4} = 4.$$

The moduli of the given numbers are, respectively,

$$|\sqrt{3} + i| = \sqrt{3 + 1} = 2,$$

and
$$|-1 + i\sqrt{3}| = \sqrt{1 + 3} = 2.$$

Hence, the product of the two moduli is the modulus of the product.

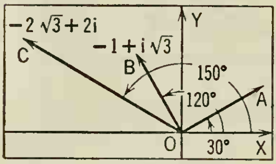


Fig. 55

From geometry, we know that the smallest angle of a right triangle whose legs are in the ratio $\frac{1}{\sqrt{3}}$ is 30°. (Prove this.) Therefore, in Figure 55, angle $XOA = 30°$, angle $XOB = 120°$, and angle $XOC = 150°$. So the angle of the product is the sum of the angles of the two factors.

Thus, the angles and moduli satisfy the relations stated in the theorem.

EXAMPLE 3. Find the modulus of the product of $3 + 4i$ and $-12 + 5i$ without multiplying together the complex numbers.

Solution: The modulus of $3 + 4i$ is

$$\sqrt{3^2 + 4^2} = 5,$$

and the modulus of $-12 + 5i$ is

$$\sqrt{(-12)^2 + 5^2} = 13.$$

Hence, the modulus of the product is

$$(5)(13) = 65. \quad \textit{Ans.}$$

The student should recall that two complex numbers which differ in the signs of their imaginary parts are called **conjugate** complex numbers. The indicated quotient of any two complex numbers may be written in a simple form by multiplying the numerator and the denominator by the conjugate of the denominator.

EXAMPLE 4. Simplify $\dfrac{1 + i\sqrt{3}}{\sqrt{2} - i}$.

Solution: Since we know that $(\sqrt{2} - i)(\sqrt{2} + i) = (\sqrt{2})^2 - i^2 = 3$, we multiply the numerator and the denominator by $\sqrt{2} + i$ to find

$$\frac{(1 + i\sqrt{3})(\sqrt{2} + i)}{(\sqrt{2} - i)(\sqrt{2} + i)} = \frac{(\sqrt{2} - \sqrt{3}) + i(\sqrt{6} + 1)}{3}$$

$$= \frac{\sqrt{2} - \sqrt{3}}{3} + \frac{\sqrt{6} + 1}{3} i. \quad \textit{Ans.}$$

The student should observe that the simplification consists in writing the quotient in the form $a + bi$, where a and b are real but not necessarily "simple" numbers.

EXERCISES 77

Simplify each of the following expressions:

1. $i^8 + i^{12}$
2. $i^9 + i^{11}$
3. $(-i)^{14} - i^{20}$
4. $(-i)^{30}$
5. $-(-i)^{15}$
6. i^{62}
7. $(-i)^{15} + i^{17}$
8. $i^{21} + i^{22} + i^{23}$
9. $-i^{11} + i^{12} + i^{13}$

Perform the indicated operations and find the modulus of the resulting complex number in each of Exercises 10 to 23.

10. $(3 - 4i)(4 + 3i)$
11. $(5 - 12i)(12 - 5i)$
12. $(2\sqrt{3} - 2i)(3 + i3\sqrt{3})$
13. $(\sqrt{3} + i)(-4\sqrt{3} - 4i)$
14. $(1 - i)^2(1 + i)$
15. $(1 + i)^3(1 - i)$
16. $\dfrac{3 + 4i}{4 - 3i}$
17. $\dfrac{8 - 6i}{1 - i\sqrt{3}}$
18. $\dfrac{8}{\sqrt{11} + i\sqrt{5}}$
19. $\dfrac{12}{\sqrt{7} - i\sqrt{2}}$
20. $\dfrac{1 - 2i}{1 + 2i} - \dfrac{1 + 2i}{1 - 2i}$
21. $\dfrac{3 - i}{3 + i} + \dfrac{3 + i}{3 - i}$
22. $\left(\dfrac{4i}{1 - i}\right)^3$
23. $\left(\dfrac{2i}{\sqrt{3} - i}\right)^3$

24. Find the value of $x^3 + 3x^2 - 16$ if $x = 2 - i$.

25. Find the value of $z^4 - z^2 + 5$ if $z = 1 - i$.

26. Show that $1 - i\sqrt{2}$ is a root of $\dfrac{1}{x - 2} - \dfrac{1}{x} = -\dfrac{2}{3}$.

27. Show that $2 + 4i$ is a root of $\dfrac{5}{x} - \dfrac{5}{x - 4} = 1$.

95. General Properties of Complex Numbers

The following properties of complex numbers are of importance in the succeeding work.

Property 1: *A complex number is zero if, and only if, the real and imaginary parts are both zero.*

Property 2: *If two complex numbers are equal, their real parts are equal, and the coefficients of their imaginary parts are equal.*

Property 3: *If the product of two complex numbers is zero, at least one of the complex numbers is zero.*

These three statements are proved as follows:

If $$a + bi = 0,$$

then, $$a = -bi.$$

Now square both sides $$a^2 = (-bi)^2 = -b^2.$$

Since a and b are real, a^2 and b^2 must both be positive or zero. If they were positive, $a^2 = -b^2$ would mean that a positive number is equal to a negative number, which is impossible. Hence, $a = b = 0$, and Property (1) is proved.

If $$a + bi = c + di,$$

then, $$a - c + i(b - d) = 0.$$

The last equation states that a certain complex number is equal to zero. Therefore, by Property (1),

$$a - c = 0 \quad \text{or} \quad a = c,$$

and $$b - d = 0 \quad \text{or} \quad b = d.$$

This proves Property (2).

If a complex number is zero, we see that its modulus must be zero.

Therefore, if $$(a + bi)(c + di) = 0,$$

then, $$|(a + bi)(c + di)| = 0.$$

But, $$|(a + bi)(c + di)| = |a + bi| \cdot |c + di|$$
$$= \sqrt{a^2 + b^2} \cdot \sqrt{c^2 + d^2},$$

where the two radicals are both real numbers. Since the product of two

real numbers cannot be zero unless at least one of them is zero, we must have

$$\sqrt{a^2 + b^2} = 0,$$

or

$$\sqrt{c^2 + d^2} = 0,$$

or both radicals are zero.

If

$$\sqrt{a^2 + b^2} = 0,$$

then,

$$a^2 + b^2 = 0,$$

and

$$a^2 = -b^2,$$

from which it follows that $a = b = 0$ and $a + bi = 0$.

If $\sqrt{a^2 + b^2} \neq 0$, then $\sqrt{c^2 + d^2} = 0$, and we could conclude that $c + di = 0$. This completes the proof of Property (3).

EXAMPLE 1. If $x + y + i(x - y + 2) = 0$, and x and y are real, find the values of x and y.

Solution: By using Property (1) above, we have

$$x + y = 0,$$

and

$$x - y + 2 = 0.$$

We find the solution of this system to be $x = -1$, $y = 1$. *Ans.*

EXAMPLE 2. For what real values of x and y is the equation $2x - 3y + i(x - 6y) = 7 + 8i$ satisfied?

Solution: By referring to Property (2) above, we have

$$2x - 3y = 7,$$

and

$$x - 6y = 8.$$

The solution of this system is $(2, -1)$. *Ans.*

EXERCISES 78

Find the real values of the letters for which each of the following equations is valid:

1. $(2x + y) + (x - 3y)i = 4 + 9i$
2. $(w - 6z) + (2w + 10z)i = -10 + 2i$
3. $(6u - 8v) - (5u - 3v)i = 29 - 15i$
4. $(5x + 3y) - (2x - y)i = 11 - 11i$

5. $(18 + 12i)u + (5i - 11)v = 3 + 39i$
6. $(5 - 7i)a + (1 - 5i)b = 1 - 14i$
7. $3c + 2d + 4 + 8i = -4ci - 6di$
8. $3(x - 2) + 4(x - 1)i = 3(y + i) + 4(y + 2)$

9. For what real values of u, v, x, and y will
$[2u + v + (u - 2v + 5)i][5x + 3y - (x + 2y + 7)i] = 0$?

10. For what real values of a, b, c, and d will
$[3a - 4b + 24 + (3a - 2b)i][c - 2d + (2c + d - 15)i] = 0$?

Complex Numbers in Trigonometric Form

NOTE: For the remainder of this chapter, it is assumed that the student has studied trigonometry.

96. The Polar Form of a Complex Number

The form $a + bi$ of a complex number is known as the **rectangular form** because, as we have seen, a and b may be taken as the rectangular components of the vector representation of the number. This graphical representation makes it possible for us to put the complex number into another useful form.

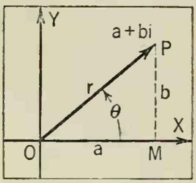


Fig. 56

The vector representation of $a + bi$ is shown in Figure 56. The length or magnitude of the vector has been labeled r, and we immediately recognize the fact that

$$r = \sqrt{a^2 + b^2} \tag{1}$$

is what we have previously called the *modulus* of the complex number. Furthermore, the angle θ in the figure is given by the equation

$$\tan\theta = \frac{b}{a}\cdot \tag{2}$$

Other names for the angle of the complex number are the *amplitude* or the *argument* of the number.

From the right triangle OMP in Figure 56, we have

$$a = r\cos\theta \quad \text{and} \quad b = r\sin\theta. \tag{3}$$

Since the angle θ is always to be measured from the positive X direction, the last two equations are correct, regardless of the quadrant in which the point P is located.

We may now write

$$a + bi = r \cos \theta + (r \sin \theta)i$$

$$= r(\cos \theta + i \sin \theta). \qquad (4)$$

This new form, which displays the modulus and the angle of the number better than the rectangular form, is called the **polar form** of the complex number.

As we know from trigonometry, the angle θ may be increased or decreased by an integral multiple of 360° without changing the value of either the sine or the cosine of the angle. Accordingly, the polar form of a number may be written

$$r[\cos (\theta + k \cdot 360°) + i \sin (\theta + k \cdot 360°)], \qquad (5)$$

where k may be zero or any positive or negative integer. This expression for a complex number is called the **complete polar form** of the number.

If a number is to be changed from the rectangular form $a + bi$ to the polar form given by (4) or (5), the relations (1) and (2) may be used. In connection with the use of relation (2), the student must be careful to obtain the proper quadrant for the angle θ. The separate signs of b and a are of importance in the determination of this quadrant, as we can see from the graphical representation of the complex number. In no case should the student change a number from rectangular to polar form without plotting the number.

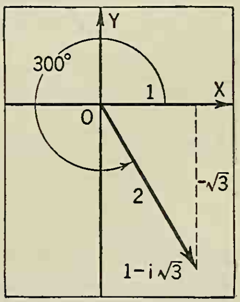


Fig. 57

EXAMPLE 1. Write the number $1 - i\sqrt{3}$ in (*a*) the polar form; (*b*) the complete polar form.

Solution: (*a*) The number $1 - i\sqrt{3}$ is plotted in Figure 57. Either from the figure, or from Equation (1), we have

$$r = \sqrt{(1)^2 + (-\sqrt{3})^2} = \sqrt{1 + 3} = 2.$$

Also from Equation (2),

$$\tan \theta = \frac{-\sqrt{3}}{1}.$$

Hence $\theta = 300°,$

and $$1 - i\sqrt{3} = 2(\cos 300° + i \sin 300°). \quad \textit{Ans.}$$

(*b*) The complete polar form is

$$\left.\begin{array}{l} 2[\cos(300° + k \cdot 360°) + i \sin(300° + k \cdot 360°)], \\ \qquad\qquad\qquad\qquad k = 0, \pm 1, \pm 2, \cdots. \end{array}\right\} \quad \textit{Ans.}$$

The student's attention is again called to the fact that in polar form r is never negative, and θ is measured in the counterclockwise sense from the positive X direction. The convention for θ implies that the signs prefixed to $\sin \theta$ and $\cos \theta$ are both *plus*. (It may be noted that the number zero has a zero modulus, but the angle is ambiguous. We shall have no occasion to use a polar form for zero.)

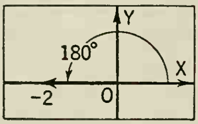


Fig. 58

Example 2. Write the number -2 in polar form.

Solution: From Figure 58, we see that $\theta = 180°$ and $r = 2$. Accordingly,

$$-2 = 2(\cos 180° + i \sin 180°). \quad \textit{Ans.}$$

Example 3. Express the number $8 - 15i$ in polar form.

Solution: The modulus of the number is

$$r = \sqrt{8^2 + (-15)^2}$$
$$= \sqrt{289} = 17.$$

Also, $$\tan \theta = \frac{-15}{8}$$
$$= -1.875;$$

and from Figure 59 it appears that θ is a fourth-quadrant angle. We use trigonometric tables to find

$$\theta = 298.1°, \text{ approximately.}$$

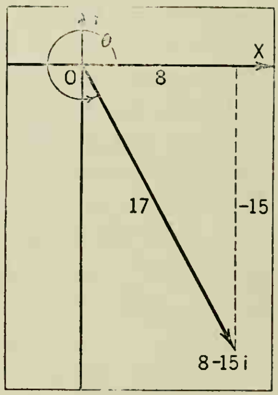


Fig. 59

Hence, $$8 - 15i = 17(\cos 298.1° + i \sin 298.1°). \quad \textit{Ans.}$$

Example 4. Express the number $10(\cos 225° + i \sin 225°)$ in rectangular form.

Solution: From trigonometry, we know that

$$\cos 225° = -\frac{\sqrt{2}}{2} \quad \text{and} \quad \sin 225° = -\frac{\sqrt{2}}{2}.$$

Therefore, $10(\cos 225° + i \sin 225°) = 10\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)$

$$= -5\sqrt{2} - i5\sqrt{2}. \quad \textit{Ans.}$$

EXERCISES 79

Write each of the following complex numbers in the polar form:

1. $-8 + 0i$ **2.** $14 + 0i$ **3.** $18i$

4. $-19i$ **5.** $3\sqrt{2} - i3\sqrt{2}$ **6.** $-11\sqrt{3} + 11i$

7. $\sqrt{15} - i\sqrt{5}$ **8.** $-4 - 4i$ **9.** $10 - i10\sqrt{3}$

10. $-13 - i13\sqrt{3}$ **11.** $\sqrt{5} + i\sqrt{15}$ **12.** $2\sqrt{6} - i2\sqrt{2}$

13. $4 - 3i$ **14.** $12 + 5i$ **15.** $-6 + 12i$

16. $6 + 8i$ **17.** $-13 - 10i$ **18.** $7 - 11i$

Write each of the following complex numbers in the complete polar form:

19. $5\sqrt{2} + i5\sqrt{2}$ **20.** $-7 - 7i$ **21.** -23

22. $31i$ **23.** $-8\sqrt{3} + 8i$ **24.** $3 - i3\sqrt{3}$

25. $-4\sqrt{3} - 4i$ **26.** $15\sqrt{2} - i15\sqrt{2}$ **27.** $10 + 20i$

Write each of the following complex numbers in the rectangular form:

28. $6(\cos 60° + i \sin 60°)$ **29.** $18(\cos 135° + i \sin 135°)$

30. $17(\cos 270° + i \sin 270°)$ **31.** $41(\cos 90° + i \sin 90°)$

32. $2\sqrt{3}(\cos 150° + i \sin 150°)$ **33.** $5\sqrt{2}(\cos 225° + i \sin 225°)$

34. $6\sqrt{2}(\cos 45° + i \sin 45°)$ **35.** $10(\cos 300° + i \sin 300°)$

36. $12(\cos 330° + i \sin 330°)$ **37.** $22(\cos 240° + i \sin 240°)$

38. $40(\cos 70° + i \sin 70°)$ **39.** $8(\cos 140° + i \sin 140°)$

40. $6(\cos 345° + i \sin 345°)$ **41.** $15(\cos 235° + i \sin 235°)$

97. Multiplication and Division of Complex Numbers in Polar Form

The multiplication and division of complex numbers is especially easy when the numbers are in polar form. We have already shown that the modulus of the product of two complex numbers is the product of their moduli, and the angle of the product is the sum of their angles. Thus, if we have the two numbers

$$a_1 + b_1 i = r_1(\cos \theta_1 + i \sin \theta_1)$$

and

$$a_2 + b_2 i = r_2(\cos \theta_2 + i \sin \theta_2),$$

then, $(a_1 + b_1 i)(a_2 + b_2 i) = r_1 r_2[\cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2)].$ (1)

This result may also be obtained by direct multiplication of the two numbers as follows:

$$[r_1(\cos \theta_1 + i \sin \theta_1)][r_2(\cos \theta_2 + i \sin \theta_2)]$$
$$= r_1 r_2[(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)].$$

The student should recognize the two quantities in parentheses in the right member as the formulas for $\cos (\theta_1 + \theta_2)$ and $\sin (\theta_1 + \theta_2)$, respectively. Alternatively, he may regard our procedure as another method for obtaining these formulas; for, if the two results obtained above are to be equal, the real parts must be equal, and the coefficients of i must be equal.

The reciprocal of a complex number $r(\cos \theta + i \sin \theta)$ may be put into polar form by multiplying numerator and denominator by the conjugate of $\cos \theta + i \sin \theta$, that is, by $\cos \theta - i \sin \theta$ as follows:

$$\frac{1}{r(\cos \theta + i \sin \theta)} = \frac{\cos \theta - i \sin \theta}{r(\cos^2 \theta + \sin^2 \theta)}.$$

Recalling that $\cos^2 \theta + \sin^2 \theta = 1$, we have

$$\frac{1}{r(\cos \theta + i \sin \theta)} = \frac{1}{r} (\cos \theta - i \sin \theta).$$

Furthermore, we know from trigonometry that $\cos (-\theta) = \cos \theta$ and $\sin (-\theta) = -\sin \theta$; hence, by replacing θ by $-\theta$ in the preceding result we obtain

$$\frac{1}{r(\cos \theta + i \sin \theta)} = \frac{1}{r} [\cos (-\theta) + i \sin (-\theta)]. \tag{2}$$

We may now obtain a useful method for the division of one complex number by another. Thus,

$$\frac{r_1(\cos \theta_1 + i \sin \theta_1)}{r_2(\cos \theta_2 + i \sin \theta_2)} = \frac{r_1}{r_2}\left(\frac{1}{\cos \theta_2 + i \sin \theta_2}\right)(\cos \theta_1 + i \sin \theta_1)$$
$$= \frac{r_1}{r_2}[\cos (-\theta_2) + i \sin (-\theta_2)][\cos \theta_1 + i \sin \theta_1]$$
$$= \frac{r_1}{r_2}[\cos (\theta_1 - \theta_2) + i \sin (\theta_1 - \theta_2)]. \tag{3}$$

This result may be stated verbally: *The modulus of the quotient of two complex numbers is the modulus of the dividend divided by the modulus of*

the divisor; the angle of the quotient is the angle of the dividend minus the angle of the divisor.

EXAMPLE 1. Obtain the product of

$$2(\cos 40° + i \sin 40°) \text{ and } 3(\cos 10° + i \sin 10°).$$

Solution:

$$[3(\cos 10° + i \sin 10°)][2(\cos 40° + i \sin 40°)]$$
$$= 6(\cos 50° + i \sin 50°). \quad \textit{Ans.}$$

EXAMPLE 2. Express $\dfrac{4(\cos 130° + i \sin 130°)}{2(\cos 40° + i \sin 40°)}$ in rectangular form.

Solution: We first carry out the division in polar form to obtain

$$2(\cos 90° + i \sin 90°).$$

Since $\cos 90° = 0$ and $\sin 90° = 1$,

we find the result to be $2i$. *Ans.*

The theorem on the product of two complex numbers leads to the following interesting and important result. In Equation (1) of this section, we put

$$r_1 = r_2 = r \quad \text{and} \quad \theta_1 = \theta_2 = \theta,$$

and find $[r(\cos \theta + i \sin \theta)]^2 = r^2(\cos 2\theta + i \sin 2\theta)$.

We may multiply both members of this equation by $r(\cos \theta + i \sin \theta)$ to obtain

$$[r(\cos \theta + i \sin \theta)]^3 = r^3(\cos 3\theta + i \sin 3\theta).$$

This result may evidently be generalized for any positive integral exponent. Therefore,

$$[r(\cos \theta + i \sin \theta)]^n = r^n(\cos n\theta + i \sin n\theta). \tag{4}$$

The statement given in Equation (4) is called **De Moivre's theorem.**

By making use of Equation (2), we may verify De Moivre's theorem for the case where n is a negative integer. Let $n = -m$, where m is a positive integer. Then,

$$[r(\cos \theta + i \sin \theta)]^{-m} = r^{-m}(\cos \theta + i \sin \theta)^{-m}$$
$$= r^{-m}\left[\frac{1}{(\cos \theta + i \sin \theta)^m}\right].$$

Since the theorem is valid for positive integral exponents, the last expression may be written

$$r^{-m}\left[\frac{1}{\cos m\theta + i \sin m\theta}\right],$$

which by (2) is equal to the expression

$$r^{-m}[\cos(-m\theta) + i \sin(-m\theta)].$$

The desired verification is therefore complete.

EXAMPLE 3. Evaluate $(\sqrt{3} + i)^6$.

Solution: Change the complex number $\sqrt{3} + i$ to polar form:

$$(\sqrt{3} + i) = 2(\cos 30^\circ + i \sin 30^\circ).$$

Hence,
$$(\sqrt{3} + i)^6 = 2^6(\cos 180^\circ + i \sin 180^\circ)$$
$$= 64(-1 + i \cdot 0) = -64. \quad \textit{Ans.}$$

EXAMPLE 4. Evaluate $(-1 - i)^{-5}$.

Solution:
$$-1 - i = 2^{1/2}(\cos 225^\circ + i \sin 225^\circ).$$

Therefore,
$$(-1 - i)^{-5} = 2^{-5/2}[\cos(-1125^\circ) + i \sin(-1125^\circ)]$$
$$= 2^{-5/2}(\cos 315^\circ + i \sin 315^\circ)$$
$$= \frac{1}{4\sqrt{2}}\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$$
$$= \frac{1}{8} - \frac{i}{8} \cdot \quad \textit{Ans.}$$

EXERCISES 80

Perform the indicated operations and express each result in the rectangular form:

1. $[2(\cos 18^\circ + i \sin 18^\circ)][4(\cos 12^\circ + i \sin 12^\circ)]$
2. $[10(\cos 34^\circ + i \sin 34^\circ)][3(\cos 26^\circ + i \sin 26^\circ)]$
3. $[7(\cos 112^\circ + i \sin 112^\circ)][2(\cos 68^\circ + i \sin 68^\circ)]$
4. $[6(\cos 223^\circ + i \sin 223^\circ)][5(\cos 227^\circ + i \sin 227^\circ)]$
5. $\dfrac{12(\cos 72^\circ + i \sin 72^\circ)}{3(\cos 42^\circ + i \sin 42^\circ)}$
6. $\dfrac{24(\cos 154^\circ + i \sin 154^\circ)}{6(\cos 64^\circ + i \sin 64^\circ)}$

7. $\dfrac{42(\cos 8^\circ + i \sin 8^\circ)}{7(\cos 68^\circ + i \sin 68^\circ)}$

8. $\dfrac{6\sqrt{2}(\cos 171^\circ + i \sin 171^\circ)}{2(\cos 216^\circ + i \sin 216^\circ)}$

9. $[2(\cos 30^\circ + i \sin 30^\circ)]^4$

10. $[4(\cos 10^\circ + i \sin 10^\circ)]^6$

11. $[3(\cos 144^\circ + i \sin 144^\circ)]^5$

12. $[2(\cos 210^\circ + i \sin 210^\circ)]^7$

13. $[\frac{1}{2}(\cos 18^\circ + i \sin 18^\circ)]^{-5}$

14. $[\frac{1}{3}(\cos 30^\circ + i \sin 30^\circ)]^{-6}$

15. $(\cos 15^\circ + i \sin 15^\circ)^{100}$

16. $(\cos 60^\circ + i \sin 60^\circ)^{50}$

17. $\left(\dfrac{\sqrt{2}}{2} - \dfrac{\sqrt{2}}{2}\, i\right)^{30}$

18. $\left(-\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}\, i\right)^{40}$

19. $(\sqrt{3} + i)^5$

20. $(\sqrt{2} - i\sqrt{2})^9$

98. Roots of Complex Numbers

As in the case of real numbers, we define an nth root of a complex number $r(\cos \theta + i \sin \theta)$ to be a complex number $R(\cos \phi + i \sin \phi)$ such that

$$[R(\cos \phi + i \sin \phi)]^n = r(\cos \theta + i \sin \theta).$$

Since n is a positive integer, we may apply De Moivre's theorem to the left side of this equation to obtain

$$R^n(\cos n\phi + i \sin n\phi) = r(\cos \theta + i \sin \theta),$$

or

$$R^n \cos n\phi + iR^n \sin n\phi = r \cos \theta + ir \sin \theta.$$

In order that this equation may be valid, the real parts must be equal, and the coefficients of i must be equal.

Thus,

$$R^n \cos n\phi = r \cos \theta,$$

and

$$R^n \sin n\phi = r \sin \theta.$$

By dividing the members of the second equation by the corresponding members of the first, we find

$$\tan n\phi = \tan \theta,$$

which is satisfied if

$$n\phi = \theta,$$

that is, if

$$\phi = \frac{\theta}{n}.$$

With this value of ϕ, we obtain

$$R^n = r,$$

or

$$R = r^{1/n}$$

These two results enable us to write

$$[r(\cos\theta + i\sin\theta)]^{1/n} = r^{1/n}\left[\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right],$$

where the exponent $1/n$ applied to the left member indicates an nth root. On the right, since r is a real, positive number, we may restrict $r^{1/n}$ to mean the principal nth root of r.

For a general fractional exponent, we have

$$\begin{aligned}[r(\cos\theta + i\sin\theta)]^{m/n} &= \{[r(\cos\theta + i\sin\theta)]^m\}^{1/n}\\ &= [r^m(\cos m\theta + i\sin m\theta)]^{1/n}\\ &= r^{m/n}\left(\cos\frac{m\theta}{n} + i\sin\frac{m\theta}{n}\right),\end{aligned}$$

a result which shows that De Moivre's theorem is valid for any rational exponent.

If we return now to the problem of nth roots and use the complete polar form of a complex number, we obtain

$$\begin{aligned}&\{r[\cos(\theta + k\cdot 360°) + i\sin(\theta + k\cdot 360°)]\}^{1/n}\\ &\quad = r^{1/n}\left[\cos\left(\frac{\theta}{n} + \frac{k}{n}\cdot 360°\right) + i\sin\left(\frac{\theta}{n} + \frac{k}{n}\cdot 360°\right)\right],\\ &\qquad \text{where } k = 0, \pm 1, \pm 2, \cdots.\end{aligned}$$

It appears from this formula that the values $k = 0, 1, 2, \cdots, (n-1)$ will yield n different values of the angle $\frac{\theta}{n} + \frac{k}{n}\cdot 360°$, differing by integral multiples of $\frac{360°}{n}$. These n values of the angle will give n different values for the desired nth root; however, any other value of k will give an angle which differs from one of these n angles by an integral multiple of 360°. In this latter case, since the sine and cosine both have periods of 360°, no additional value for the nth root will be obtained.

The preceding discussion is summarized as follows: *Every complex number* $r(\cos\theta + i\sin\theta)$, $r \neq 0$, *has exactly* n *distinct* n*th roots. These roots all have the same modulus, the positive number* $r^{1/n}$. *The angles of these* n*th roots may be taken as*

$$\frac{\theta}{n}, \frac{\theta}{n} + \frac{1}{n}\cdot 360°, \frac{\theta}{n} + \frac{2}{n}\cdot 360°, \cdots, \frac{\theta}{n} + \frac{n-1}{n}\cdot 360°,$$

respectively.

EXAMPLE 1. Find the four 4th roots of $-8 + i8\sqrt{3}$.

Solution: We first write the given number in polar form.

$$-8 + i8\sqrt{3} = 16(\cos 120° + i \sin 120°).$$

We now find $16^{1/4} = 2, \quad \dfrac{120°}{4} = 30°, \quad$ and $\quad \dfrac{360°}{4} = 90°.$

Hence, the required 4th roots all have the modulus 2, and the angles may be taken as

$$30°, 30° + 90°, 30° + 2 \cdot 90°, \text{ and } 30° + 3 \cdot 90°,$$

respectively. (We have used $k = 0, 1, 2,$ and 3.) The four 4th roots are listed below.

$$\left.\begin{aligned} k = 0&: \quad 2(\cos 30° + i \sin 30°) = \sqrt{3} + i; \\ k = 1&: \quad 2(\cos 120° + i \sin 120°) = -1 + i\sqrt{3}; \\ k = 2&: \quad 2(\cos 210° + i \sin 210°) = -\sqrt{3} - i; \\ k = 3&: \quad 2(\cos 300° + i \sin 300°) = 1 - i\sqrt{3}. \end{aligned}\right\} \quad \textit{Ans.}$$

Notice that the value $k = 4$ would yield

$$2(\cos 390° + i \sin 390°) = 2(\cos 30° + i \sin 30°),$$

a repetition of the result for $k = 0$.

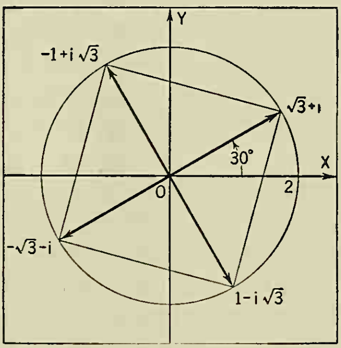


Fig. 60

The four roots just obtained are represented graphically in Figure 60. The four points corresponding to the roots are the vertices of a square inscribed in a circle of radius 2 units and with center at the origin.

It is evident from the general discussion of this section that the points

representing the nth roots of $r(\cos\theta + i\sin\theta)$ will lie at the vertices of a regular polygon of n sides inscribed in a circle of radius $r^{1/n}$ and with center at the origin. The angle $\dfrac{\theta}{n}$ is the angle of that one of the roots which has the smallest positive (or zero) angle.

EXERCISES 81

Find in the polar form the roots of the equation in each of Exercises 1 to 16. Represent the roots of each equation graphically on one diagram.

1. $y^2 = 36(\cos 80° + i\sin 80°)$

2. $x^2 = 4(\cos 140° + i\sin 140°)$

3. $v^3 = 27(\cos 72° + i\sin 72°)$

4. $z^3 = 8(\cos 105° + i\sin 105°)$

5. $x^4 = 81(\cos 64° + i\sin 64°)$

6. $y^4 = 16(\cos 200° + i\sin 200°)$

7. $w^5 = \cos 150° + i\sin 150°$

8. $v^6 = 27(\cos 120° + i\sin 120°)$

9. $x^2 = 1 + i\sqrt{3}$

10. $w^2 = 8 - i8\sqrt{3}$

11. $y^3 + 4 + i4\sqrt{3} = 0$

12. $z^3 - 8\sqrt{2} - i8\sqrt{6} = 0$

13. $v^4 = 2 - i2\sqrt{3}$

14. $x^4 = -8 + i8\sqrt{3}$

15. $s^5 - 32 = 0$

16. $y^5 + 16\sqrt{3} + 16i = 0$

Find all the roots of each of the equations listed below, and express these roots in both the polar and the rectangular form. Represent the roots of each equation graphically on one diagram.

17. $w^2 + 36i = 0$

18. $x^2 = 32 + i32\sqrt{3}$

19. $y^3 - 27 = 0$

20. $v^3 - 8i = 0$

21. $x^3 + 216 = 0$

22. $z^3 + 27i = 0$

23. $v^3 - 64i = 0$

24. $t^4 + 4 = 0$

25. $x^4 + 81 = 0$

26. $y^4 = -32 - i32\sqrt{3}$

27. $y^6 - 64 = 0$

28. $x^6 + 8 = 0$

29. $x^5 - 1 = 0$

30. $u^5 - 243i = 0$

Chapter 16

INEQUALITIES

99. Definitions

On several occasions we have made statements to the effect that one number is greater than another, or that one number is less than another. Such statements are referred to as **inequalities.** For example, $2 > -1$ and $5 < 7$ are inequalities.

Since a statement of inequality between two numbers implies that the numbers may be arranged in order of magnitude on a number scale, it follows that inequalities in the algebraic sense must deal with real numbers only. This restriction must be understood throughout algebra.

As we have seen from our previous work, it takes very definite knowledge to describe a number by means of an equation. When, as it may happen, we have less definite knowledge about a number than equations require, it may be that our information is sufficient to describe the number by means of an inequality. Thus, the statement $x - 1 = 4$ describes the number x completely; x must have the value 5. The statement $x - 1 > 4$ describes x in a less definite fashion; x may have any value greater than 5.

The student is again reminded that the symbols $>$ and $<$ always point toward the smaller number. Two inequalities in which the inequality signs point in the same direction are said to have the "same sense"; if the signs point in opposite directions, the inequalities are "opposite in sense." Thus, the statements $a > b$ and $c > d$ have the same sense; $a > b$ and $c < d$ are opposite in sense.

We sometimes use the symbols $\geqq$ and $\leqq$ to mean "is greater than or equal to" and "is less than or equal to," respectively. For example, every real number, say N, obeys the inequality $N^2 \geqq 0$.

As in the case of equations, inequalities are divided into two kinds, defined as follows:

(1) An **unconditional** or **absolute** inequality is an inequality that is valid for all permissible values of the letters involved. For example, the statement $a^2 + 4 > 0$, which is valid for every real value of a, is an absolute inequality.

(2) A **conditional** inequality is an inequality that is valid only for a certain set or range of real values of the letters involved. Thus, $x - 1 > 0$ is a conditional inequality; it is valid only if $x > 1$.

100. Operations with Inequalities

The methods used in handling inequalities are in many ways similar to those used in dealing with equations. The following theorems are all proved by the use of the fundamental fact that if

$$a > b,$$

then, $\quad a - b = p$ is a *positive* number,

and conversely.

Theorem 1: *The addition of the same real number to, or the subtraction of the same real number from, both members of an inequality leaves the sense of the inequality unaltered.*

Proof: The inequality $a > b$ means $a - b = p$ is a positive number. For any real number m, we have

$$a - b = a + m - b - m$$

$$= (a + m) - (b + m).$$

If we substitute the last expression for $a - b$ into the preceding equation, we obtain

$$(a + m) - (b + m) = p.$$

Therefore, if $\quad a > b,$

then, $\quad a + m > b + m.$

Since m may be positive or negative, Theorem (1) is proved.

Illustrations: If we start with the inequality $6 > 2$, we may find $11 > 7$ by adding 5 to both members. Similarly, $-1 > -5$ is obtained by subtracting 7 from both members.

Theorem 2: *The multiplication or division of both members of an inequality by the same positive number leaves the sense of the inequality unaltered.*

Proof: If $a > b$, we have $a - b = p$ is a positive number. If q is any positive number, then,

$$aq - bq = pq,$$

which is also positive. Hence, if

$$a > b \quad \text{and} \quad q > 0,$$

then,
$$aq > bq.$$

Theorem 3: *If both members of an inequality are multiplied or divided by a negative number, a new inequality of opposite sense is obtained.*

The proof of Theorem (3) is left for the student. It follows from Theorem (3) that, if the signs of both members of an inequality are reversed, the sense of the inequality will also be reversed.

Illustrations: From the inequality $3 < 5$, we find $6 < 10$ by multiplying both members by 2. Also, we may obtain $-6 > -10$ by multiplying both members by -2.

Theorem 4: *If all the members of two inequalities of the same sense are positive and the corresponding members of the inequalities are multiplied together, an inequality of the same sense is obtained.*

Proof: If $a > b$ and $c > d$, with a, b, c, and d all positive, we have

$$a - b = p \quad \text{and} \quad c - d = q,$$

with p and q positive. Therefore,

$$a = b + p \quad \text{and} \quad c = d + q.$$

By multiplying together the corresponding members of these two equations, we find

$$ac = bd + bq + dp + pq$$

or
$$ac - bd = bq + dp + pq$$

which is positive. (Why?) Hence, if $a > b$, $c > d$, and a, b, c, and d are positive,

$$ac > bd.$$

Illustration: From the inequalities $5 > 2$ and $4 > 3$ we may conclude $20 > 6$.

It follows from Theorem (4) that *if both members of an inequality are positive, any positive powers of both members are unequal in the same sense;* that is, from

$$a > b, \quad (a \text{ and } b \text{ positive})$$

we have

$$a^p > b^p. \quad (p \text{ positive})$$

The proof of this is left to the student: first, for the case where p is a positive integer; second, for the case where $p = 1/n$, where n is a positive integer; and, third, for the case where $p = m/n$, where m and n are positive integers. For the second case, the method of "reduction to an absurdity" might be employed.

Illustration: Since $6 > 5$, then, $6^2 > 5^2$; $\sqrt{6} > \sqrt{5}$; and $6^{2/3} > 5^{2/3}$. It must be understood that only principal roots are used in the last two inequalities.

101. Unconditional Inequalities

Many important properties of numbers may be expressed in the form of inequalities. In this section we shall deal briefly with unconditional inequalities. A proposed unconditional inequality may frequently be shown to be valid by a process of analysis as follows:

(1) Assume the inequality to be valid.

(2) Apply any of the properties (1) to (4) of the preceding section which may be necessary in order to deduce from the assumed statement another inequality which is known to be valid.

(3) Reverse the steps in (2), making certain that each step is justified, in order to obtain the proposed inequality. The student who is familiar with the principles of logic will realize that step (3) is essential; the demonstration cannot stop with step (2).

Example 1. Prove that the sum of any positive number and its reciprocal cannot be less than 2, that is,

$$a + \frac{1}{a} \geqq 2, \qquad \text{if } a > 0.$$

Analysis: Assume the validity of the proposed inequality. Since $a > 0$, we may multiply both sides of $a + 1/a \geqq 2$ by a to obtain

$$a^2 + 1 \geqq 2a. \qquad \text{[Theorem (2)]}$$

Next, subtract $2a$ from both sides to get

$$a^2 - 2a + 1 \geqq 0, \qquad \text{[Theorem (1)]}$$

or

$$(a - 1)^2 \geqq 0.$$

The last statement is valid since the square of a real number must be positive or zero.

Proof: Thus, to demonstrate the desired proposition, let us start with

$$(a - 1)^2 \geqq 0$$

as a known, valid inequality. By expanding the left side, we may write

$$a^2 - 2a + 1 \geqq 0.$$

Add $2a$ to both members to get

$$a^2 + 1 \geqq 2a. \qquad \text{[Theorem (1)]}$$

Since $a > 0$, we may divide both members by a to find

$$a + \frac{1}{a} \geqq 2. \qquad \text{[Theorem (2)]}$$

This proves the validity of the proposed inequality.

Example 2. Show that the arithmetic mean is always greater than the positive geometric mean of two unequal positive numbers, that is,

$$\frac{a+b}{2} > \sqrt{ab}, \quad \text{if } a > 0, \quad b > 0, \quad \text{and} \quad a \neq b.$$

Analysis: If

$$\frac{a+b}{2} > \sqrt{ab},$$

then,

$$a + b > 2\sqrt{ab}. \qquad \text{[Theorem (2)]}$$

Also

$$a - 2\sqrt{ab} + b > 0, \qquad \text{[Theorem (1)]}$$

or

$$(\sqrt{a} - \sqrt{b})^2 > 0,$$

which is a correct statement, since $a \neq b$.

Proof of the Given Proposition: Since

$$a \neq b,$$

then,

$$\sqrt{a} \neq \sqrt{b}$$

and

$$\sqrt{a} - \sqrt{b} \neq 0.$$

Therefore,

$$(\sqrt{a} - \sqrt{b})^2 > 0,$$

or

$$a - 2\sqrt{ab} + b > 0.$$

Hence,

$$a + b > 2\sqrt{ab}$$

and $$\frac{a+b}{2} > \sqrt{ab}.$$

This is the desired inequality. The statement that a and b are both positive has entered implicitly in the assumption that only real numbers are involved.

EXAMPLE 3. Without expressing any roots decimally, show that

$$\sqrt{41} + \sqrt{39} > \sqrt{42} + \sqrt{38}.$$

Analysis: If the required inequality is assumed to be correct and both sides are squared, there is obtained

$$41 + 2\sqrt{1599} + 39 > 42 + 2\sqrt{1596} + 38.$$

Hence, $$2\sqrt{1599} > 2\sqrt{1596},$$

or $$\sqrt{1599} > \sqrt{1596}$$

and $$1599 > 1596.$$

Proof: The last inequality above is known to be correct. The steps in the analysis may be taken in reverse order if only principal square roots are used. Hence, the proposed inequality is correct.

EXERCISES 82

Show the validity of each of the following inequalities:

1. $\dfrac{b}{3a} + \dfrac{3a}{b} > 2$, if $a > 0$, $b > 0$, $3a \neq b$
2. $\dfrac{3d}{4c} > 1 - \dfrac{c}{3d}$, if $c > 0$, $d > 0$, $2c \neq 3d$
3. $\dfrac{h^2}{2k^2} \geqq 1 - \dfrac{k^2}{2h^2}$, if $h \neq 0$, $k \neq 0$
4. $\dfrac{\sqrt{u}}{\sqrt{v}} + \dfrac{\sqrt{v}}{\sqrt{u}} > 2$, if $u > 0$, $v > 0$, $u \neq v$
5. $e^2 + f(13f + e) > 3f(e + 4f)$, if $e \neq f$
6. $\dfrac{x + 2y}{4x} > \dfrac{2y}{x + 2y}$, if $x > 0$, $x + 2y > 0$, $x \neq 2y$
7. $4 + \dfrac{1}{4z^2} \geqq \dfrac{2}{z}$, if $z \neq 0$
8. $a^2 + \dfrac{16}{a^2} \geqq 8$, if $a \neq 0$

9. $\dfrac{1}{x^3} + \dfrac{1}{8y^3} > \dfrac{1}{2x^2y} + \dfrac{1}{4xy^2}$, if $x > 0,\ y > 0,\ x \neq 2y$

10. $\dfrac{2c^4d^4}{c^8 - d^8} < \dfrac{c^2d^2}{c^4 - d^4}$, if $c > d$, $cd \neq 0$

11. $j^2 + k^2 + m^2 > jk + km + mj$, unless $j = k = m$

12. $(ab + cd)(ac + bd) \geqq 4abcd$, if $a > 0,\ b > 0,\ c > 0,\ d > 0$

13. Show that $2 \geqq rs + tu$ if $r^2 + 4s^2 = 4$ and $t^2 + 4u^2 = 4$

14. If a, b, and c are positive numbers and $\dfrac{d}{a} < \dfrac{e}{b} < \dfrac{f}{c}$, show that

$$\frac{d}{a} < \frac{d + e + f}{a + b + c} < \frac{f}{c}.$$

Show that each of the following inequalities is correct. Do not use approximate roots in any step.

15. $\sqrt{11} + \sqrt{17} < \sqrt{15} + \sqrt{13}$

16. $\sqrt{23} - \sqrt{7} > \sqrt{17} - \sqrt{13}$

17. $\sqrt{2} - \sqrt{14} + \sqrt{17} - \sqrt{5} < 0$

18. $\sqrt{67} < \sqrt{5} + \sqrt{7} + \sqrt{11}$

19. If $|e| \neq \sqrt{2}|f|$, which is greater, $4e^2f^2 - 10f^4$ or $e^4 - 6f^4$?

20. If $|r| \neq |s|$, which is greater, $r^4 - 5s^4$ or $2r^2s^2 - 6s^4$?

102. Conditional Inequalities

We shall confine the following discussion entirely to conditional inequalities involving one letter. An inequality is said to be solved when the range of values of the letter which makes the inequality valid is

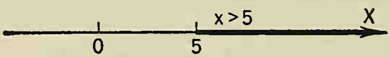


Fig. 61

found. This range of values is called the **solution** of the inequality. For example, the inequality

$$2x - 1 > x + 4$$

may be solved by adding 1 to both sides and subtracting x from both sides to obtain

$$x > 5.$$

The range of values of x greater than 5 is the solution of the inequality. This range may be indicated as in Figure 61.

In the next sections we shall consider two methods of solution of

conditional inequalities, namely, the algebraic method and the graphical method.

103. The Algebraic Solution of Inequalities

An inequality which contains the unknown to the first degree only is called a *linear* inequality. Linear inequalities and inequalities that may be reduced to this type are easily solved by algebraic manipulation as in the preceding section.

EXAMPLE 1. Find the range of values for which $5x + 6 > 8x + 12$.

Solution:

Subtract $8x$ from both members $\quad -3x + 6 > +12.$

Add -6 to both members $\quad -3x > 6.$

Fig. 62

Divide both sides by -3 and reverse the sense according to Theorem (3), Section 100, to get

$$x < -2. \quad \textit{Ans.}$$

This range is shown in Figure 62.

By means of algebraic manipulation, an inequality may always be put into the form

$$f(x) > 0 \quad \text{or} \quad f(x) < 0.$$

If $f(x)$ is a quadratic function $ax^2 + bx + c$, the inequality may be solved by the method of completing the square. Essentially, this method is based on the fact that

$$x^2 > n^2, \qquad (n > 0)$$

is valid if, and only if,

$$|x| > n;$$

that is, $\quad x > n \quad \text{or} \quad x < -n \quad$ (see Figure 63*a*).

By contrast, the inequality

$$x^2 < n^2, \qquad (n > 0)$$

is valid if, and only if,

$$|x| < n;$$

that is, both $x > -n$ and $x < n$ are to be satisfied. This last condition is written conveniently in the form

$$-n < x < n,$$

which is read "x lies between $-n$ and n" (see Figure 63b). Note that two inequalities may be combined in this manner only when the variable

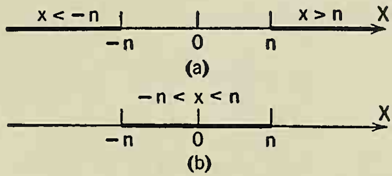


Fig. 63

lies between two numbers. It may not be done when the range is broken, as in Figure 63a.

EXAMPLE 2. Solve the inequality $x^2 + 4x + 3 \geqq 0$.

Solution:

Subtract 3 from both members $\quad x^2 + 4x \geqq -3.$

In order to have a perfect trinomial square on the left, we add 4 to both sides. This addition yields

$$x^2 + 4x + 4 \geqq 1,$$

or

$$(x + 2)^2 \geqq 1.$$

Fig. 64

By referring to the discussion in the preceding paragraph, we see that we must have

$$x + 2 \geqq 1 \qquad \text{or} \qquad x + 2 \leqq -1,$$

that is, $\qquad x \geqq -1 \qquad$ or $\qquad x \leqq -3.$ *Ans.*

This range is displayed in Figure 64.

EXAMPLE 3. For what values of x is $2x^2 + 6x < 9$?

Solution:

Divide both sides by 2 $\qquad x^2 + 3x < \frac{9}{2}$.

In order to obtain a perfect square on the left, add $(\frac{3}{2})^2 = \frac{9}{4}$ to both members

$$x^2 + 3x + \tfrac{9}{4} < \tfrac{27}{4},$$

or
$$(x + \tfrac{3}{2})^2 < \tfrac{27}{4}.$$

The last inequality is valid if

$$\left|x + \frac{3}{2}\right| < \frac{3\sqrt{3}}{2},$$

or
$$-\frac{3\sqrt{3}}{2} < x + \frac{3}{2} < \frac{3\sqrt{3}}{2}.$$

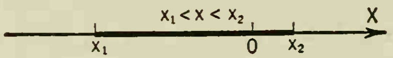


Fig. 65

If $\frac{3}{2}$ is subtracted from each member of the last statement, we obtain

$$-\frac{3\sqrt{3}}{2} - \frac{3}{2} < x < \frac{3\sqrt{3}}{2} - \frac{3}{2},$$

or
$$-\frac{3}{2}(\sqrt{3} + 1) < x < \frac{3}{2}(\sqrt{3} - 1). \quad \textit{Ans.}$$

See Figure 65 in which

$$x_1 = -\frac{3}{2}(\sqrt{3} + 1) \qquad \text{and} \qquad x_2 = \frac{3}{2}(\sqrt{3} - 1).$$

EXAMPLE 4. Solve the inequality $4x^2 - 4x + 5 < 0$.

Solution:

Subtract 5 from both sides $\qquad 4x^2 - 4x < -5$.

Divide both sides by 4 $\qquad x^2 - x < -\frac{5}{4}$.

Add $(\frac{1}{2})^2 = \frac{1}{4}$ to both sides $\qquad x^2 - x + \frac{1}{4} < -1$,

or $\qquad (x - \frac{1}{2})^2 < -1$.

Since $(x - \frac{1}{2})^2$ cannot be negative for any real value of x, the given inequality is not valid for any real values of x. *Ans.*

A rational integral function of x is often called a **polynomial in x.** Thus, $f(x) = 3x^3 - 5x^2 + 4x + 7$ is a polynomial in x. If the function in the inequalities $f(x) > 0$ and $f(x) < 0$ is a polynomial in factored form, the inequalities may be solved as in the following example.

EXAMPLE 5. Solve the inequality $(x + 1)(x - 2)(x - 3) < 0$.

Solution: The product $(x + 1)(x - 2)(x - 3)$ is negative if all three factors are negative or if two are positive and the third is negative. This means that

$$(x + 1)(x - 2)(x - 3) < 0$$

is valid for any one of the following situations:

(*a*) $x + 1 < 0$, $x - 2 < 0$, and $x - 3 < 0$;

(*b*) $x + 1 < 0$, $x - 2 > 0$, and $x - 3 > 0$;

(*c*) $x + 1 > 0$, $x - 2 < 0$, and $x - 3 > 0$;

(*d*) $x + 1 > 0$, $x - 2 > 0$, and $x - 3 < 0$.

In each case, the common range of values (if there is one) of the three linear inequalities gives a solution of the original inequality. The situation for each of these four cases may be displayed in a manner similar to that employed in Figure 66 for case (*a*).

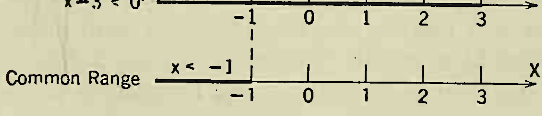


Fig. 66

The student may show similarly that the inequalities in (*b*) and (*c*) have no common range of validity and that those in (*d*) have the common range $2 < x < 3$. Thus, the given inequality is valid for

$$x < -1 \qquad \text{or} \qquad 2 < x < 3. \quad \textit{Ans.}$$

Although the fundamental ideas used in Example 5 should be clearly understood, the method itself is too cumbersome to be recommended. The graphical solution which is discussed in the next sections is much more efficient in this kind of problem.

EXERCISES 83

Solve each of the following inequalities:

1. $5x - 6 < 19$

2. $3z - 14 > 7$

3. $6y - 10 > 2y - 22$

4. $11h + 10 > 14h + 31$

5. $(x - 3)(x + 4) < (x + 5)(x - 2)$

6. $(2u - 6)(u + 1) < (u - 4)(2u + 3)$

7. $y^2 - 6y + 8 < 0$

8. $x^2 - x - 12 < 0$

9. $18 - 7s - s^2 \geqq 0$

10. $35 - 2v - v^2 \leqq 0$

11. $8 < 18v + 5v^2$

12. $15 > 12m^2 + 8m$

13. $m^2 - 3m - 2 > 0$

14. $7y^2 \geqq 12y - 3$

15. $w(4w + 6) < 1$

16. $x(2x + 5) < 5$

17. $-7 < 4y(y - 1)$

18. $v(v - 6) > -13$

19. $3z(3z - 4) \leqq 4$

20. $w^2 - 8w + 20 > 0$

21. $\dfrac{20}{y^2} > 3 + \dfrac{7}{y}$

22. $\dfrac{35}{x^2} > \dfrac{41}{x} - 12$

For what range of values of k will each of the following expressions be imaginary?

23. $\sqrt{3k + 11}$

24. $\sqrt{12k - 27}$

25. $\sqrt{6k^2 - 19k + 10}$

26. $\sqrt{8k^2 - 10k - 7}$

27. $\sqrt{24k - 18 - 9k^2}$

28. $\sqrt{-23 + 20k - 4k^2}$

104. The Graph of a Factored Polynomial

In order to prepare for the graphical solution of inequalities that can be reduced to the problem of finding the range of values for which a polynomial in x is positive (or negative), we shall consider here the problem of sketching rapidly the graph of such a function. For the remainder of this section, $f(x)$ will be used to denote a polynomial in factored form.

Figure 67 shows the graphs of the functions

(a) $\qquad f(x) = x - k;$

(b) $\qquad f(x) = (x - k)^2;$

(c) $\qquad f(x) = (x - k)^3,$

plotted for values of x close to the value $x = k$. In each case, the value $x = k$ gives a zero value for the function; hence, the graph has the point $(k, 0)$ in common with the X axis.

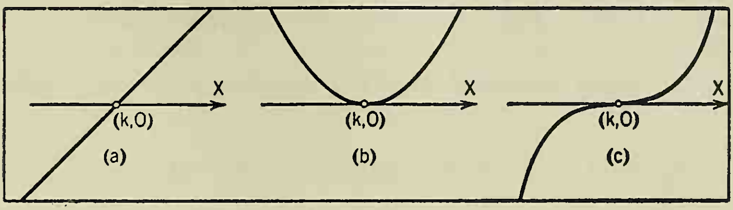


Fig. 67

In (a) and (c), $f(x)$ changes sign as x changes from values less than k to values greater than k. Thus, the curves cross the X axis at $(k, 0)$. On the other hand, the function in (b) is a second power; it cannot have any negative values. Hence, the curve in Figure 67*b* remains above the X axis except at $(k, 0)$ where it touches (is tangent to) the axis.

It should be evident that

$$f(x) = (x - k)^n, \qquad \text{if} \qquad n = 5, 7, 9, \cdots,$$

will have a graph that is quite similar to that in Figure 67*c*, and that,

$$f(x) = (x - k)^n, \qquad \text{if} \qquad n = 4, 6, 8, \cdots,$$

will have a graph which resembles that in Figure 67*b*. The actual change in appearance caused by increasing the value of the integral exponent n consists in a "flattening" of the curve close to the point $(k, 0)$.

If three different factors of the type just discussed are combined in a single product, the function will have a graph showing all three of the general forms displayed in Figure 67. For example, Figure 68 shows the graph of

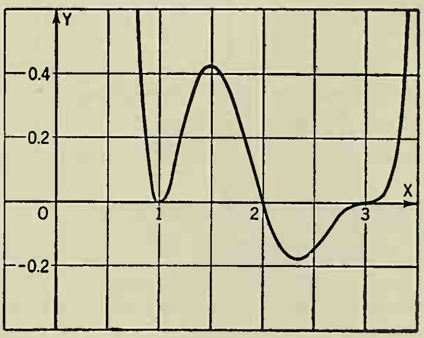


Fig. 68

$$f(x) = (x - 1)^2(x - 2)(x - 3)^3.$$

At $(1, 0)$ we find the typical intersection for a squared factor; at $(2, 0)$

for a linear factor; and at (3, 0) for a cubic factor. That this appearance of the graph is as it should be may easily be seen by noting that for values of x less than 2, the factors $(x - 2)$ and $(x - 3)^3$ cannot change sign and, hence, cannot affect the general way in which the curve crosses the axis at (1, 0). A similar statement holds for the points (2, 0) and (3, 0).

The preceding discussion may be generalized to fit any function of the form

$$f(x) = A(x - a_1)^{p_1}(x - a_2)^{p_2}(x - a_3)^{p_3} \cdots (x - a_n)^{p_n},$$

where $p_1, p_2, p_3, \cdots, p_n$ are positive integers. The graph of $f(x)$ will meet the X axis at the points $(a_1, 0)$, $(a_2, 0)$, $(a_3, 0)$, $\cdots$, $(a_n, 0)$. At each such point the curve will cross in the general manner of a straight line if the corresponding exponent is 1 (Figure 67*a*) and somewhat like the cubic if the exponent is odd and greater than 1 (Figure 67*c*). If the exponent is even, the curve touches the axis in the general manner of a second power (Figure 67*b*). The detailed shape of the curve between points where the curve touches or crosses the X axis is not of importance here and may be indicated schematically. In order to find whether the curve lies above or below the axis for values of x greater than the largest a, it may be observed that for such values all of the binomial factors are

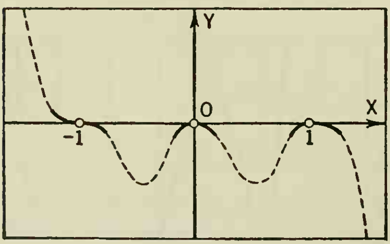


Fig. 69

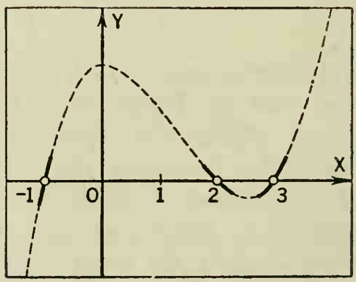


Fig. 70

positive; hence, the sign of $f(x)$ is determined by the sign of the coefficient A.

Example 1. Make a schematic drawing showing how the graph of the function $f(x) = -6x^2(x + 1)^5(x - 1)^4$ meets the X axis.

Solution: Since $f(x)$ is zero for $x = 0, -1, 1$, the curve meets the X axis at the corresponding points. The factor x has the exponent 2;

hence, at (0, 0) the curve touches, but does not cross, the X axis. The factor $(x + 1)$ bears the exponent 5; therefore, the curve crosses the horizontal axis at $(-1, 0)$ in the general manner of Figure 67*c*. Finally, the factor $(x - 1)$ has the exponent 4, and the curve touches, but does not cross, the axis at (1, 0). For values of x greater than 1, $f(x)$ will have negative values. This fact results from the sign of the coefficient -6 since each of the other factors of $f(x)$ is positive for $x > 1$. The required schematic graph is shown in Figure 69.

EXERCISES 84

Sketch a schematic graph for each of the following polynomial functions:

1. $f(x) = (x - 4)(x + 1)(x + 5)$ **2.** $f(z) = (z - 2)(z - 3)(z + 6)$
3. $f(w) = w(7 + w)(4 - w)$ **4.** $f(v) = (v + 3)(2 - v)(v + 5)$
5. $f(y) = (y + 4)(y - 3)^2$ **6.** $f(u) = (5 - u)^2(4 - u)$
7. $f(x) = (x - 2)^2(x + 2)^3$ **8.** $f(p) = p^4(2p - 5)$
9. $f(k) = 3(k - 3)^4(3k + 7)$ **10.** $f(x) = -4x^5(6 + 9x)^3$
11. $f(s) = (s - 2)(s + 3)(5 + s)(4 - s)$
12. $f(x) = x^2(x - 4)^2(2 + x)(6 - x)$
13. $f(v) = v^3(v - 1)^3(v - 4)^4(v + 3)^5$
14. $f(y) = (y - 2)^3(5 - y)^5(2 + y)^4(8 - y)^2$

105. Graphical Solution of Inequalities

The graphical solution of inequalities such as $f(x) > 0$ or $f(x) < 0$ consists of two steps, as follows:

(1) Draw the graph of the function $f(x)$.

(2) Keeping in mind that $f(x) > 0$ [that is, $f(x)$ is *positive*] when its graph lies *above* the X axis, and $f(x) < 0$ [$f(x)$ is *negative*] when its graph lies *below* the X axis, read the range of values of x for which the given inequality is valid.

An inequality may frequently be reduced to one of the preceding forms, where $f(x)$ is a factored polynomial. The schematic sketch of such a polynomial may then be used to read off the solution of the inequality.

EXAMPLE 1. Solve the inequality $(x + 1)(x - 2)(x - 3) < 0$ by the graphical method. (Compare Example 5 of Section 103.)

Solution: Let $f(x) = (x + 1)(x - 2)(x - 3)$.

The schematic graph of $f(x)$ is given in Figure 70. Since it is required

to have $f(x) < 0$, we read off the ranges of values of x for which the curve lies below the X axis. These ranges are

$$x < -1 \qquad \text{or} \qquad 2 < x < 3. \quad \textit{Ans.}$$

EXAMPLE 2. For what values of x is $\dfrac{x(x+2)^2}{x-1} > 0$?

Solution: Since we do not know at first whether $x - 1$ is a positive or negative number, we multiply the numerator and the denominator of the given fraction by $x - 1$ in order to obtain an even power in the denominator. This gives

$$\frac{x(x+2)^2(x-1)}{(x-1)^2} > 0.$$

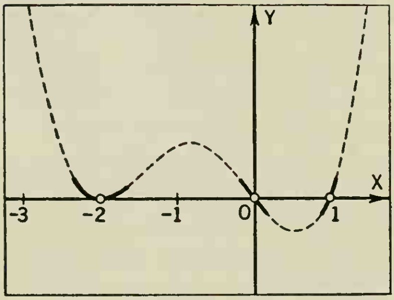


Fig. 71

Since $x = 1$ is not a permissible value, $(x - 1)^2$ is a positive number for any value that x may assume; hence, the multiplication of both sides of the inequality by $(x - 1)^2$ will not change the sense. This multiplication gives

$$x(x+2)^2(x-1) > 0$$

Now, let $f(x) = x(x + 2)^2(x - 1)$, and draw the schematic graph of $f(x)$ as in Figure 71. The required values of x are those permissible values for which the curve lies above the X axis. Hence,

$$x < -2, \qquad -2 < x < 0, \qquad \text{or} \qquad x > 1. \quad \textit{Ans.}$$

EXAMPLE 3. Find the values of r for which $\left|\dfrac{r}{4-r}\right| < 1$.

Solution: Since the *numerical* value of the fraction $\dfrac{r}{4-r}$ is to be less than 1, we have to find those values of r for which

$$-1 < \frac{r}{4-r} < 1.$$

This means that we must determine the range of values of r for which the two conditions

$$\frac{r}{4-r} < 1 \tag{1}$$

and
$$\frac{r}{4-r} > -1 \tag{2}$$
are both satisfied.

In order to obtain the solution of this system of inequalities, we first multiply both sides of (1) by $(4 - r)^2$, a positive quantity $(r \neq 4)$, to get
$$r(4 - r) < (4 - r)^2.$$
This inequality is equivalent to
$$r(4 - r) - (4 - r)^2 < 0,$$
or
$$-2(r - 2)(r - 4) < 0.$$
Upon dividing both sides by -2, we find
$$(r - 2)(r - 4) > 0. \tag{3}$$

Similarly, we obtain from (2) the inequality
$$r(4 - r) > -(4 - r)^2,$$
which is equivalent to
$$r(4 - r) + (4 - r)^2 > 0,$$
$$4(4 - r) > 0,$$
or
$$4 - r > 0. \tag{4}$$

The common range of validity of (3) and (4) is the required solution and may be obtained graphically by sketching the two functions
$$F(r) = (r - 2)(r - 4), \qquad \text{and} \qquad f(r) = 4 - r,$$
on the same diagram (Figure 72). The range of values to be read is the one for which both graphs lie above the horizontal axis. Thus, we find that the given inequality is satisfied for $r < 2$. *Ans.*

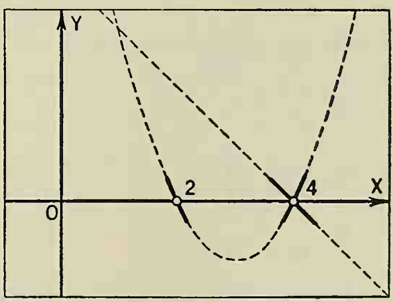


Fig. 72

When solving a system of two inequalities, as in the last example, it is convenient to have both inequalities written in the same sense so that we may read from the graph the range where both curves lie above the X axis (or where both lie below the axis).

It should be noted that, had the inequality in Example 3 been $\left|\dfrac{r}{4-r}\right| > 1$, we would have had to find the values of r for which

$$\frac{r}{4-r} < -1 \qquad \text{or} \qquad \frac{r}{4-r} > 1.$$

In this case, the inequalities are to be satisfied separately, not simultaneously.

EXERCISES 85

Solve each of the following inequalities by the graphical method:

1. $y^2 + 2y - 35 > 0$
2. $z^2 + 9z - 10 > 0$
3. $2x^2 + x - 21 < 0$
4. $6v^2 - 11v - 10 < 0$
5. $32 + 12u - 5u^2 \leqq 0$
6. $77 - 23x - 12x^2 \leqq 0$
7. $4 > x(x - 7)$
8. $5 < y(y + 2)$
9. $5w^2 + w - 13 \geqq 0$
10. $3x^2 - 2x - 11 \leqq 0$
11. $(y - 5)(y + 7)(y - 2) < 0$
12. $(u + 3)(u + 7)(u + 11) > 0$
13. $(5 - 2m)(4 - m)(2 + m) > 0$
14. $(3 + h)(6 - h)(7 - 2h) < 0$
15. $(x - 3)^3(x + 7) < 0$
16. $(5z - 8)(z - 4)^3 < 0$
17. $(r + 1)(r - 3)(r + 7)(r - 8) \leqq 0$
18. $(t + 6)(t - 4)(t - 1)(t + 3) \geqq 0$
19. $(y - 4)^3(y + 7)^2(y - 2)^4 < 0$
20. $(s + 5)(s - 2)^3(s + 8)^2 > 0$
21. $\dfrac{5}{z + 12} < 0$
22. $\dfrac{13}{2k - 15} > 0$
23. $\dfrac{v + 5}{v - 11} < 0$
24. $\dfrac{x - 4}{x - 8} > 0$
25. $\dfrac{(y - 3)^2}{y - 2} < 0$
26. $\dfrac{(2u + 7)^2}{3u - 10} > 0$
27. $\dfrac{m + 6}{(m + 2)(m - 2)^3} > 0$
28. $\dfrac{w + 3}{(w + 8)^4(w - 12)} < 0$
29. $\dfrac{v + 5}{v - 2} > \dfrac{v - 4}{v + 1}$
30. $\dfrac{x + 7}{x + 4} > \dfrac{x + 5}{x + 3}$
31. $\left|\dfrac{1}{y + 8}\right| < 1$
32. $\left|\dfrac{1}{k - 10}\right| > 1$
33. $\left|\dfrac{x}{x + 12}\right| < 1$
34. $\left|\dfrac{2v}{v + 12}\right| < 1$

For what range of values of x will each of the following systems of inequalities be valid?

35. $6x - x^2 > 0$, $x^2 - 2x > 0$
36. $x^2 - 3x - 4 < 0$, $2x - x^2 < 0$

37. $5 - 4x - x^2 < 0$
$x^3 - 16x < 0$

38. $x^3 - 25x > 0$
$x^2 - 9x > 0$

For what range of values of y will each of the following radicals be real and not zero?

39. $\sqrt{(3 + y)(7 + y)(8 - y)}$

40. $\sqrt{(4 - y)(9 - y)(2 + y)}$

41. $\sqrt{(y + 4)^5(y - 3)^3}$

42. $\sqrt{(2y - 9)(y^2 - 25)}$

43. $\sqrt{(y^2 - 6y + 10)(6 - y^2)}$

44. $\sqrt{(y^2 - 4y + 1)(16 - y^2)}$

106. Inequalities and Quadratic Equations

Besides the problems previously considered, there are a number of other types of problems in which the coefficients of a quadratic equation are to be so determined as to satisfy certain requirements with respect to the roots. For example, it may be required to determine the range of values of k in the equation

$$kx^2 + 12x + 9k = 0$$

so that the roots are real and unequal.

The required determination may be accomplished by using the fact that the discriminant must be positive in order for the equation to have real, unequal roots. Hence, for the given equation,

$$\begin{aligned} b^2 - 4ac &= (12)^2 - 4(k)(9k) \\ &= 144 - 36k^2 \end{aligned}$$

must be positive. This means that we need the solution of the inequality $144 - 36k^2 > 0$. The student may show by the methods of the preceding sections that the required range of values of k is $-2 < k < 2$. As a partial verification of this result, the value $k = 1$ gives the equation $x^2 + 12x + 9 = 0$, which has the real, unequal roots $-6 + 3\sqrt{3}$ and $-6 - 3\sqrt{3}$.

EXAMPLE 1. For what range of values of m will the equation

$$2x^2 + 2mx + 1 = mx^2$$

have imaginary roots? real roots?

Solution: The given equation may be written in standard quadratic form as follows:

$$(2 - m)x^2 + 2mx + 1 = 0.$$

The discriminant is

$$\begin{aligned} b^2 - 4ac &= 4m^2 - 4(2 - m) \\ &= 4(m^2 + m - 2). \end{aligned}$$

In order to satisfy the requirement that the given equation have imaginary roots, we must make the discriminant negative, that is,

$$4(m^2 + m - 2) < 0,$$

or

$$m^2 + m - 2 < 0,$$

and

$$(m + 2)(m - 1) < 0.$$

Using the methods of the preceding sections, we find

$$-2 < m < 1. \quad \textit{Ans.}$$

The student may show similarly that the discriminant will be nonnegative and hence the original equation will have real roots if

$$m \leqq -2 \qquad \text{or} \qquad m \geqq 1. \quad \textit{Ans.}$$

Example 2. For what values of k will the parabola given by the equation $y = x^2 - 2kx + k$ cut the X axis in two distinct points? be tangent to the X axis? lie entirely above the X axis?

Solution: The curve represented by the given equation will cut the X axis in two points, be tangent to the axis, or lie entirely above it according as the equation $x^2 - 2kx + k = 0$ has real and unequal, real and equal, or imaginary roots. Hence, we must find the discriminant. It is

$$\begin{aligned} b^2 - 4ac &= 4k^2 - 4k \\ &= 4k(k - 1). \end{aligned}$$

An investigation of this quadratic function in k reveals the following:

The discriminant is positive, that is, the parabola will intersect the X axis in two distinct points, if

$$k < 0 \qquad \text{or} \qquad k > 1. \quad \textit{Ans.}$$

The discriminant is zero, which means that the parabola will be tangent to the axis, if

$$k = 0 \qquad \text{or} \qquad k = 1. \quad \textit{Ans.}$$

The discriminant is negative, which implies that the parabola will lie entirely above the axis, if

$$0 < k < 1. \quad \textit{Ans.}$$

EXERCISES 86

For each of Equations 1 to 4, determine for what values of m the roots are real and unequal.

1. $y^2 - my + 36 = 0$
2. $mx^2 + (2m - 1)x + 3 = 0$
3. $(m - 2)v^2 + 2mv + 9 = 0$
4. $(m + 3)z^2 = m(z - 1) + 3$

For each of Equations 5 to 14, determine for what values of k or m the roots are real and unequal; real and equal; imaginary.

5. $y^2 - my + 9 = 0$
6. $z^2 + 5kz + 6k^2 + 4 = 0$
7. $(2 - k)x^2 - 2kx - 3k = 0$
8. $2mv^2 + 4v + m + 1 = 0$
9. $m^2w^2 + mw = 2m - 1$
10. $2k + 1 = 2x(x - k) - x$
11. $4x^2 + 4mx + 2m^2 - m - 12 = 0$
12. $(k + 2)z^2 + 2kz - k - 1 = 0$
13. $(m - 6)x^2 + (x - 2)m^2 = 0$
14. $k^2y^2 + k^2y = 3y - 1$

For each of the following problems, determine the values of k so that the graph of the equation (a) cuts the X axis in two distinct points; (b) is tangent to the X axis; (c) does not meet the X axis:

15. $y = x^2 + kx + k + 3$
16. $y = 2kx^2 - 2kx + k - 50$
17. $y = kx^2 - 7x + k$
18. $y = 3x(x - 2k) + k + 10$

Chapter 17

THEORY OF EQUATIONS

107. Rational Integral Equations

Many of the important problems of algebra involve equations of the form

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0, \tag{1}$$

where n is a positive integer and the a's are constants. Equation (1) is called a **rational integral equation of the nth degree in x**; the function on the left side of this equation is called a **polynomial of the nth degree in x.** Throughout this chapter the symbol $P(x)$ will denote such a polynomial. Thus, (1) may be written

$$P(x) = 0. \tag{2}$$

Illustration: The equation $5x^3 - 47x + 63 = 0$ is a cubic (third degree) equation in x. The coefficients are $a_0 = 5$, $a_1 = 0$, $a_2 = -47$, and $a_3 = 63$. Notice that, corresponding to the degree n, there are $n + 1$ coefficients; missing powers of x are supplied with zero coefficients. In this illustration $P(x) = 5x^3 - 47x + 63$.

We have previously considered the simple cases where $n = 1$ (linear equations) and $n = 2$ (quadratic equations). Corresponding to the values $n = 3$ and $n = 4$, respectively, the adjectives *cubic* and *quartic* are often used.

One of the most important problems in connection with a rational integral equation is that of finding its roots. We shall discuss certain fundamental theorems and describe methods for finding real roots of equations with given numerical coefficients. General formulas for the solution of the cubic and the quartic equations will be given.

108. Synthetic Division

In much of the work of this chapter it will be necessary to divide a polynomial $P(x)$ by a binomial $x - r$. We shall see in this section how a division of this kind may be put into an abbreviated form.

We may regard the form which follows in (1) as illustrative of an ordinary long division. This form shows the division of

$$3x^3 + 5x^2 - 7x + 10 \quad \text{by} \quad x - 2.$$

$$
\text{(1)} \qquad
\begin{array}{rrrrrl}
 & 3x^2 & +11x & +15 & & \text{(Quotient)} \\
\cline{2-5}
\text{(Dividend)} & 3x^3 & +5x^2 & -7x & +10 & \big|\; x - 2 \quad \text{(Divisor)} \\
 & 3x^3 & -6x^2 & & & \\
\cline{2-3}
 & & +11x^2 & & & \\
 & & +11x^2 & -22x & & \\
\cline{3-4}
 & & & +15x & & \\
 & & & +15x & -30 & \\
\cline{4-5}
 & & & & +40 & \text{(Remainder)}
\end{array}
$$

If we omit the powers of x and write only the coefficients in their proper positions, we have the same division appearing in (2). Notice the repetition of the starred numbers in the first three columns.

$$
\text{(2)} \qquad
\begin{array}{rrrrrl}
 & 3^* & +11^* & +15^* & & \text{(Quotient)} \\
\cline{2-5}
\text{(Dividend)} & 3^* & +5 & -7 & +10 & \big|\; 1 - 2 \quad \text{(Divisor)} \\
 & 3^* & -6 & & & \\
\cline{2-3}
 & & +11^* & & & \\
 & & +11^* & -22 & & \\
\cline{3-4}
 & & & +15^* & & \\
 & & & +15^* & -30 & \\
\cline{4-5}
 & & & & +40 & \text{(Remainder)}
\end{array}
$$

We may take advantage of the fact that the first coefficient of the divisor is 1 by agreeing to keep this in mind rather than writing it; then we can obtain the simple form (3) if we omit the unnecessary repetitions above.

$$
\text{(3)} \qquad
\begin{array}{rrrr|r}
3 & +5 & -7 & +10 & -2 \\
 & -6 & -22 & -30 & \\
\hline
\mathbf{3} & \mathbf{+11} & \mathbf{+15} & \mathbf{+40} &
\end{array}
$$

The boldface numbers are the important items in the result. The

numbers 3, 11, and 15 are the coefficients of the quotient, and 40 is the remainder. It is convenient to make one more slight modification. In form (3) we have subtracted the second line from the first. We avoid this subtraction by changing the sign of the indicated "divisor," that is, we use $+2$ in place of -2; then, we have only to add at each step. The final form of the division is displayed in (4) below. This abbreviated form of long division is called **synthetic division.**

$$(4) \qquad \begin{array}{rrrr|l} 3 & +5 & -7 & +10 & \mathbf{2} \\ & +6 & +22 & +30 & \\ \hline \mathbf{3} & \mathbf{+11} & \mathbf{+15} & \mathbf{+40} & \end{array}$$

In summary, the process of synthetic division which is used for dividing a polynomial $P(x)$ by a binomial $x - r$ may be carried out as follows:

(1) *Write down the coefficients of the dividend, including signs, in order of descending powers of* x. *Be sure to supply a zero for each missing power.*

(2) *If the divisor is* x − r, *write* r *for the indicated divisor. Then write the first coefficient of the dividend in the first place in the third line.*

(3) *Follow the dotted arrows in form* (4). *Each number at an arrow head is obtained by multiplying the number at the tail by the indicated divisor. Each number at the tail of an arrow is obtained by adding the two numbers above it.*

(4) *When the division has been completed, the last number in the third line is the remainder. The other numbers in the same line are the coefficients of the quotient in order of descending powers of* x, *starting from the left. It should be clear that the highest power of* x *in the quotient is one less than that in the dividend.*

Example 1. Divide $2x^4 - 3x^2 + 5x - 7$ by $x + 3$.

Solution: The divisor in this example is to be regarded as $x - (-3)$. The zero in the first line of the synthetic division is put in the place of the missing power of x.

$$\begin{array}{rrrrr|l} 2 & 0 & -3 & +5 & -7 & -3 \\ & -6 & +18 & -45 & +120 & \\ \hline 2 & -6 & +15 & -40 & +113 & \end{array}$$

The quotient is $2x^3 - 6x^2 + 15x - 40$ and the remainder is 113. *Ans.*

Example 2. Divide $2x^4 + 5x^3 + 3x^2 + 4x + 8$ by $2x + 3$.

Solution: We may use synthetic division in this problem by first

writing the divisor in the form $2(x + \frac{3}{2})$. Since division by $2(x + \frac{3}{2})$ is equivalent to division in succession by $x + \frac{3}{2}$ and 2, we proceed as follows:

$$\begin{array}{rrrrr|l} 2 & +5 & +3 & +4 & +8 & -\frac{3}{2} \\ & -3 & -3 & 0 & -6 & \\ \hline 2 & +2 & 0 & +4 & +2 & \end{array}$$

This result means that

$$\frac{2x^4 + 5x^3 + 3x^2 + 4x + 8}{x + \frac{3}{2}} = 2x^3 + 2x^2 + 4 + \frac{2}{x + \frac{3}{2}}.$$

Hence, if we divide both sides by 2, we obtain

$$\frac{2x^4 + 5x^3 + 3x^2 + 4x + 8}{2x + 3} = x^3 + x^2 + 2 + \frac{2}{2x + 3}. \quad \textit{Ans.}$$

EXAMPLE 3. Show by synthetic division that 4 is a root of the equation $x^3 + 6x^2 - 15x - 100 = 0$.

Solution: If 4 is a root of the equation, $x - 4$ must be a factor of the left side, and conversely. Consequently, the problem is equivalent to showing that if the left side of the equation is divided by $x - 4$, the remainder is zero. The division is shown next.

$$\begin{array}{rrrr|l} 1 & +6 & -15 & -100 & 4 \\ & +4 & +40 & +100 & \\ \hline 1 & +10 & +25 & 0 & \end{array}$$

Thus, the remainder is zero, and 4 is a root of the equation.

EXERCISES 87

In each of Equations 1 to 4, state the value of $n, a_0, a_1, a_2, \cdots, a_n$; then, by dividing both members by the coefficient of the highest power of the unknown, change the equation to a form in which the new $a_0 = 1$.

1. $7x^4 - 5x^3 + 15x - 2 = 0$
2. $\frac{1}{2}v^5 + \frac{3}{2}v^4 - v^2 + 3 = 0$
3. $\sqrt{3}y^7 - 4y^5 + 2\sqrt{3}y - 7 = 0$
4. $-\frac{3}{4}x^8 + \frac{1}{2}x^6 - \dfrac{\sqrt{5}}{2}x^3 + 2x - 5 = 0$

In each of the following exercises, use synthetic division to find the quotient and the reminder:

5. $(v^3 - 8v - 3) \div (v - 3)$

6. $(y^3 - 4y^2 - 25) \div (y - 5)$
7. $(x^3 + 4x^2 - 7x + 5) \div (x - 2)$
8. $(4w^3 - w^2 + 92) \div (w + 3)$
9. $(z^3 + 32z + 24) \div (z + 6)$
10. $(2y^4 - 3y^3 + y^2 - 3y - 6) \div (y - 2)$
11. $(3x^4 - 41x^2 - 13x - 8) \div (x - 4)$
12. $(v^5 - 4v^3 + 5v^2 - 5) \div (v + 1)$
13. $(z^4 + 4z^3 - 9z^2 - 8z + 14) \div (z - \sqrt{2})$
14. $(2s^4 + s^3 - 11s^2 - 3s + 15) \div (s + \sqrt{3})$
15. $(x^4 - ax^3 + 3a^2x^2 - 7a^3x + 2a^4) \div (x - a)$
16. $(2y^4 + 5by^3 + 6b^2y^2 + 3b^3y - 6b^4) \div (y + 2b)$
17. $(2v^3 + 3v^2 + 2v + 3) \div (v - i) \qquad (i^2 = -1)$
18. $(3w^3 - w^2 + 12w - 4) \div (w + 2i)$
19. $(6x^3 - 19x^2 - 11x + 14) \div (2x - 7)$
20. $(12y^3 + 17y^2 + 4y + 15) \div (3y + 5)$
21. $(15z^3 - 2z^2 - 21z + 6) \div (5z + 1)$
22. $(8x^3 - 2x^2 - 10x + 17) \div (4x - 3)$
23. Show that -4 is a root of $5y^3 + 18y^2 - y + 28 = 0$.
24. Show that 6 is a root of $7x^3 - 39x^2 - 26x + 48 = 0$.
25. Show that 5 is a root of $3v^4 - 14v^3 - 7v^2 + 21v - 55 = 0$.
26. Show that -8 is a root of $3w^4 + 22w^3 - 19w^2 - 22w + 16 = 0$.

109. The Remainder Theorem

In the preceding chapter we have assumed that the graph of an equation $y = P(x)$, where $P(x)$ is a polynomial in x, is a continuous curve; that is, the value of y changes gradually if the value of x is changed gradually, so that there is no break in the curve. This fact is of importance to us here, and may be justified as follows:

If
$$P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n,$$

and any fixed value of x is denoted by x_1, we have

$$P(x_1) = a_0x_1^n + a_1x^{n-1} + \cdots + a_n.$$

Now, let the value of x be changed from x_1 to $x_1 + h$. Then,

$$P(x_1 + h) = a_0(x_1 + h)^n + a_1(x_1 + h)^{n-1} + \cdots + a_n.$$

Hence, $P(x_1 + h) - P(x_1)$ is a polynomial in x_1 and h, and every term contains h as a factor. The student can easily verify this latter statement by considering a particular polynomial. In general, the fact may be confirmed by expanding the terms of $P(x_1 + h)$ by the binomial

theorem and noting that each term that does not involve h is duplicated in $P(x_1)$. Therefore, by making h sufficiently small, we can make the difference between $P(x_1)$ and $P(x_1 + h)$ as small as we please. Thus, the value of $P(x)$ will change gradually as x is changed gradually, and the graph of $y = P(x)$, where $P(x)$ is *any* polynomial, will be a continuous curve.

Now let a polynomial $P(x)$ be divided by a binomial $x - r$, where r is a constant, and let the division be carried to the point where the remainder is a constant. If the quotient is denoted by $Q(x)$ and the remainder by R, we have

$$P(x) = (x - r)Q(x) + R.$$

(The dividend equals the remainder plus the product of the divisor and the quotient.)

This equation is valid for all values of x except possibly the value $x = r$, in view of the division by $x - r$ that led to the relationship. However, the equation must be valid for $x = r$ also. For, both members are polynomials and are therefore continuous for all values of x. Consequently, if they were not equal for $x = r$, one or the other would have a break in its continuity at this point which is impossible. Thus we may put $x = r$ to obtain

$$P(r) = (r - r)Q(r) + R.$$

Now, $Q(r)$ is a constant, and $r - r = 0$; hence,

$$\boldsymbol{P(r) = R.}$$

This discussion may be summarized in the following statement which is basic for a good deal of the ensuing work:

The Remainder Theorem: *If a polynomial* P(x) *is divided by a binomial* x − r, *and the division is carried to the point where the remainder does not involve* x, *the remainder is the value of the polynomial when* x *is replaced by* r.

Illustration: If we divide $P(x) = x^3 + 7x^2 - 9x - 10$ by $x - 5$, we have a remainder of 245 as shown below.

$$\begin{array}{rrrr|l} 1 & +7 & -9 & -10 & 5 \\ & +5 & +60 & +255 & \\ \hline 1 & +12 & +51 & \mathbf{+245} & \end{array}$$

Furthermore,

$$\begin{aligned} P(5) &= 5^3 + 7(5^2) - 9(5) - 10 \\ &= 125 + 175 - 45 - 10 = \mathbf{245}. \end{aligned}$$

Thus, as we expected, $P(5)$ is the value of the remainder when $P(x)$ is divided by $x - 5$. We see from this illustration how the remainder theorem may be used to calculate values of $P(x)$ corresponding to given values of x.

A corollary of the remainder theorem is of aid in finding factors of $P(x)$. This corollary, which is known as the **factor theorem,** states: *If the value of* P(x), *when* x *is replaced by* r, *is zero, then* x − r *is a factor of* P(x). The proof of this statement follows immediately from the remainder theorem; for, if $x = r$ results in $P(r) = 0$, the remainder is zero when $P(x)$ is divided by $x - r$; therefore, $P(x)$ is exactly divisible by $x - r$.

Values of x which result in the value zero for a function $f(x)$ are called **zeros** of the function. In other words, the zeros of $f(x)$ are the roots of the equation $f(x) = 0$.

Example 1. Show by means of the factor theorem that $x + 2$ is a factor of the polynomial $P(x) = 2x^3 + 5x + 26$.

Solution: $P(-2) = 2(-2)^3 + 5(-2) + 26$

$$= -16 - 10 + 26 = 0.$$

Hence, by the factor theorem, $x + 2$ is a factor of $P(x)$. *Ans.*

Check: By synthetic division we have

$$\begin{array}{rrrr|r} 2 & 0 & +5 & +26 & -2 \\ & -4 & +8 & -26 & \\ \hline 2 & -4 & +13 & 0 & \end{array}$$

Therefore, $x + 2$ is one factor of $2x^3 + 5x + 26$; the other factor is $2x^2 - 4x + 13$.

Example 2. Use the factor theorem to show that $x^5 - a^5$ is exactly divisible by $x - a$.

Solution: Let $P(x) = x^5 - a^5$.

Then, $P(a) = a^5 - a^5 = 0$.

Consequently, $P(x)$ is exactly divisible by $x - a$, and the quotient may be found by synthetic division.

Example 3. One zero of the function $x^3 - 8x^2 + 7$ is $x = 1$. Find the other zeros.

Solution: By the factor theorem, if $x = 1$ is a zero of the given poly-

nomial, $x - 1$ is a factor. Consequently, the other factor may be found by division:

$$\begin{array}{rrrr|r} 1 & -8 & 0 & +7 & 1 \\ & +1 & -7 & -7 & \\ \hline 1 & -7 & -7 & 0 & \end{array}$$

The division shows that $x - 1$ is a factor and that $x^2 - 7x - 7$ is the second factor of $x^3 - 8x^2 + 7$. If we write

$$x^2 - 7x - 7 = 0,$$

we find

$$x = \frac{7 \pm \sqrt{77}}{2}$$

as the other zeros. *Ans.*

EXERCISES 88

In each of Examples 1 to 4, show (*a*) by the remainder theorem and (*b*) by the use of synthetic division that the first expression is a factor of the second.

1. $v + 2;\quad v^3 - 11v - 14$

2. $x - 4;\quad 3x^3 - 10x^2 - 15x + 28$

3. $y - 6;\quad 2y^4 - 13y^3 + 6y^2 + 5y - 30$

4. $w + 5;\quad 4w^4 + 20w^3 - w^2 - 2w + 15$

By the use of synthetic division find

5. $P(3)$ and $P(-2)$, if $P(x) = 5x^3 - 6x^2 + x - 20$

6. $P(-3)$ and $P(4)$, if $P(v) = 3v^4 - 4v^3 + 10v - 11$

7. $P(6)$ and $P(-\frac{1}{2})$, if $P(w) = 4w^4 - 27w^3 + 13w - 21$

8. $P(\frac{1}{2})$ and $P(-5)$, if $P(y) = 2y^5 + y^4 - 3y^3 + 85y^2 + 18$

Use the given zero to find the remaining zeros of each of the following functions:

9. $z^3 + 6z^2 - 6z - 136$; one zero is 4.

10. $3y^3 + 13y^2 - 57y - 7$; one zero is -7.

11. $10x^3 - 23x^2 + 18x - 9$; one zero is $\frac{3}{2}$.

12. $12s^3 - 17s^2 - 6s + 8$; one zero is $\frac{2}{3}$.

13. Show by the remainder theorem that $x + 2a^2$ is a factor of $x^5 + 32a^{10}$.

14. Show by the remainder theorem that $x + 3y$ is a factor of $x^3 + 4x^2y + 4xy^2 + 3y^3$.

15. Show by the remainder theorem that $3x - 4y$ is a factor of $3x^3 - 7x^2y + xy^2 + 4y^3$.

110. The Number of Roots

As we have seen in connection with quadratic equations, not every rational integral equation has a real root. If we permit the coefficients of such an equation to be real or imaginary, it is far from evident that the equation will have any root. Consequently, the following theorem is of fundamental importance.

Theorem 1: *Every rational integral equation* $P(x) = 0$ *has at least one root.*

As the proof of this statement is considerably beyond the scope of this book, we shall accept its validity with only a brief discussion. The coefficients of $P(x)$ are not restricted to be real numbers, and the root may be real or imaginary. The theorem does not imply that formulas similar to the quadratic formula exist for higher degree equations; in fact, there are such formulas only for the solution of cubic and quartic equations. The theorem merely asserts the existence of a root but does not tell how to find it. Because of the basic character of this assertion, Theorem (1) is known as the **fundamental theorem of algebra.**

On the basis of the fundamental theorem, we can now derive a second important theorem.

Theorem 2: *Every rational integral equation* $P(x) = 0$ *of the* n*th degree has exactly* n *roots.*

Proof: Let $P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n,$

and let r_1 denote a root whose existence is guaranteed by Theorem (1). Then, by the factor theorem, $x - r_1$ is a factor of $P(x)$. Consequently,

$$P(x) = (x - r_1)P_1(x),$$

where $P_1(x)$ is a polynomial of degree $n - 1$ in x.

Now consider the equation

$$P_1(x) = 0,$$

which by the fundamental theorem also has a root. Let r_2 denote this root; then, $x - r_2$ is a factor of $P_1(x)$, and

$$P_1(x) = (x - r_2)P_2(x),$$

where $P_2(x)$ is a polynomial of degree $n - 2$ in x. Therefore, the original polynomial may be written

$$P(x) = (x - r_1)(x - r_2)P_2(x).$$

If this procedure is used n times, we obtain $P(x)$ in the factored form

$$P(x) = (x - r_1)(x - r_2) \cdots (x - r_n)P_n(x),$$

where $P_n(x)$ must be of degree $n - n$ or zero; that is, $P_n(x)$ is a constant, and by comparison with

$$P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n,$$

we find $$P_n(x) = a_0,$$

and $$P(x) = a_0(x - r_1)(x - r_2) \cdots (x - r_n).$$

It follows now that the equation $P(x) = 0$ has n roots; for $P(x)$ has the value zero if x is put equal to any of the values $r_1, r_2, \cdots, r_n$.

Furthermore, the equation $P(x) = 0$, where $P(x)$ is of the nth degree, can have no more than n roots. For, if it did, the polynomial $P(x)$ would have more than n first-degree factors, one factor corresponding to each root. Since the product of these factors would be of higher degree than n, the hypothesis that $P(x)$ is of the nth degree would be contradicted.

In the case of the quadratic equation, we have seen that an equation may have equal roots. It should be noted in the general situation that two or more of the roots $r_1, r_2, \cdots, r_n$ may be equal. If the same root occurs twice, it is called a **double** root; if three times, a **triple** root; and if m times, a root of **multiplicity m.**

By making use of Theorem (2), we may form rational integral equations with given roots; for, corresponding to the roots $r_1, r_2, \cdots, r_n$, the left side of the equation must be

$$P(x) = a_0(x - r_1)(x - r_2) \cdots (x - r_n),$$

where $a_0 \neq 0$. Hence, the equation

$$(x - r_1)(x - r_2) \cdots (x - r_n) = 0$$

will have the desired roots $r_1, r_2, \cdots, r_n$.

Example 1. Form a quartic equation which has the roots $1 - i$, $1 + i$, and 2 as a double root.

Solution: The equation in factored form is

$$[x - (1 - i)][x - (1 + i)](x - 2)(x - 2) = 0.$$

Since the left side may be written

$$[(x - 1)^2 - i^2](x - 2)^2 = (x^2 - 2x + 2)(x^2 - 4x + 4),$$

we find as the desired equation

$$x^4 - 6x^3 + 14x^2 - 16x + 8 = 0. \quad \textit{Ans.}$$

A corollary of Theorem (2) that is of importance in a number of applications of algebra is as follows:

Theorem 3: *If two polynomials in the same variable and of degree not greater than* n *are equal for more than* n *distinct values of the variable, the coefficients of like powers of the variable in the two polynomials are equal, that is, the polynomials are identical.*

Proof: Let the two polynomials be

$$a_0x^n + a_1x^{n-1} + \cdots + a_n \quad \text{and} \quad b_0x^n + b_1x^{n-1} + \cdots + b_n.$$

Then, by hypothesis, the equation

$$a_0x^n + a_1x^{n-1} + \cdots + a_n = b_0x^n + b_1x^{n-1} + \cdots + b_n$$

or

$$(a_0 - b_0)x^n + (a_1 - b_1)x^{n-1} + \cdots + (a_n - b_n) = 0$$

is of not more than the nth degree and is satisfied by *more* than n distinct values of x.

By Theorem (2) this is impossible, and it follows that all the coefficients of the last equation are zero. For, by hypothesis, the degree of the equation does not exceed n, and we have seen that the degree cannot equal n, so we must have

$$a_0 - b_0 = 0 \qquad \text{or} \qquad a_0 = b_0.$$

Upon applying the same reasoning, we see that the degree must be less than $n - 1$, so

$$a_1 - b_1 = 0 \qquad \text{or} \qquad a_1 = b_1.$$

In a similar way, we find

$$\begin{aligned} a_2 &= b_2, \\ a_3 &= b_3, \\ &\cdots\cdots \\ a_{n-1} &= b_{n-1}. \end{aligned}$$

Finally, we arrive at the equation

$$a_n - b_n = 0 \qquad \text{or} \qquad a_n = b_n,$$

and we see that the polynomials are identical.

Example 2. Determine the values of the constants A, B, and C so that $x - 1 = A(x^2 + 2) + x(Bx + C)$ is an identity.

Solution: The given equation may be written in the form

$$0 \cdot x^2 + x - 1 = (A + B)x^2 + Cx + 2A,$$

which displays the fact that we have two polynomials of not more than the second degree which are to be identical, that is, which are to be equal for more than two values of x. Therefore, by Theorem (3), we may equate corresponding coefficients to obtain

$$0 = A + B,$$

$$1 = C,$$

and

$$-1 = 2A.$$

This system of equations has the solution $A = -\frac{1}{2}$, $B = \frac{1}{2}$, $C = 1$. *Ans.*

The method used in Example 2 is often called the **method of undetermined coefficients.**

EXERCISES 89

In each of Examples 1 to 8 form the rational integral equation of lowest degree which has the given numbers for its roots. In each case, let 1 be the coefficient of the term of highest degree.

1. $-3, 1, 2, 5$

2. $-2, 4, -2 + \sqrt{3}, -2 - \sqrt{3}$

3. $-5, 0, 3, i\sqrt{2}, -i\sqrt{2}$

4. $-3, 0, 4, i\sqrt{5}, -i\sqrt{5}$

5. $1 + \sqrt{5}$, $1 - \sqrt{5}$, and -2 as a double root

6. $-4 + i\sqrt{3}$, $-4 - i\sqrt{3}$, and 3 as a double root

7. $\sqrt{2}$, $-\sqrt{2}$, and $2i$ and $-2i$ both as double roots

8. $i\sqrt{7}$, $-i\sqrt{7}$, and $\sqrt{2}$ and $-\sqrt{2}$ both as double roots

In each of Exercises 9 to 12, the given equation is to be an identity in x. Determine the value of each of the other letters to satisfy this condition.

9. $3x + 29 = A(x + 3) + B(x - 2)$

10. $x^2 - 6x - 16 = (Ax + B)(x - 4) + C(x^2 + x + 4)$

11. $3x^3 + 3x^2 + 5x - 4 = A(x^3 - 5) + (Bx^2 + Cx + D)(x + 2)$

12. $x^3 - 16x^2 + 65x + 26 = A(x - 3)(x^2 + 4) + B(x^2 + 4) + (Cx + D)(x - 3)^2$

13. The equation $z^4 - 9z^3 + 16z^2 + 15z + 25 = 0$ has 5 as a double root; find the other roots.

14. The equation $x^4 + 4x^3 - 18x^2 - 80x - 32 = 0$ has -4 as a double root; find the other roots.

15. Two roots of the equation $v^4 + 4v^3 - 4v^2 - 40v - 33 = 0$ are -1 and 3; find the others.

16. Two roots of the equation $2y^4 - 3y^3 - 41y^2 + 104y - 80 = 0$ are -5 and 4; find the others.

111. Rational Roots; Positive and Negative Real Roots

We shall now prove a theorem that is useful in finding the rational roots (if there are any) of a rational integral equation with integral coefficients.

Theorem: *Let* $\frac{p}{q}$, *a real rational fraction in its lowest terms, be a root of the equation*

$$P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0 \qquad (a_0 \neq 0)$$

where $a_0, a_1, \cdots, a_{n-1}, a_n$ *are integers. Then,* p *is a factor of* a_n, *the constant term, and* q *is a factor of* a_0, *the coefficient of* x^n.

Proof: The case where $p = 0$ is readily disposed of by observing that $x = 0$ cannot be a root of the equation unless the term $a_n = 0$. Although this may be considered a special case of the theorem, it is simpler to agree to divide $P(x)$ by any power of x that may be a factor of $P(x)$. Of course, if x^m is a factor of $P(x)$, $x = 0$ is a root of the equation of multiplicity m. We assume in the remaining discussion that $a_n \neq 0$.

If $\frac{p}{q}$ is a root of $P(x) = 0$, we must have

$$a_0\left(\frac{p}{q}\right)^n + a_1\left(\frac{p}{q}\right)^{n-1} + \cdots + a_{n-1}\left(\frac{p}{q}\right) + a_n = 0. \qquad (1)$$

If both members of Equation (1) are multiplied by q^n, the result is

$$a_0p^n + a_1p^{n-1}q + \cdots + a_{n-1}pq^{n-1} + a_nq^n = 0. \qquad (2)$$

Now, if $p = 1$, it is of course a factor of a_n and part of the theorem would be true. If $p \neq 1$, we divide both sides of Equation (2) by p to obtain

$$a_0p^{n-1} + a_1p^{n-2}q + \cdots + a_{n-1}q^{n-1} + \frac{a_nq^n}{p} = 0. \qquad (3)$$

Since p, q, and all the a's are assumed to be integers, it follows that all terms but the last on the left side of (3) are whole numbers. However, the last term cannot be a fraction, for the sum of a set of whole numbers

and a single fraction cannot be zero. Furthermore, $\frac{p}{q}$ was assumed to be in its lowest terms so that q is not divisible by p. Hence for $\frac{a_nq^n}{p}$ to be a whole number, a_n must be divisible by p.

The argument to show that q is a factor of a_0 proceeds in the same manner. If $q = 1$, it is a factor of a_0. If $q \neq 1$, we may divide both members of Equation (2) by q and complete the proof in the same way as in the preceding paragraph.

An important special case of the theorem on rational roots is the following statement:

If a rational integral equation with integral coefficients has an integer for a root, this root must be a factor of the constant term.

EXAMPLE 1. Find a rational root of the following equation, and then find the remaining roots: $2x^3 + x^2 - 2x - 6 = 0$.

Solution: If $\frac{p}{q}$ is a rational root of the equation, p must be a factor of -6, and q must be a factor of 2. We list the possibilities below:

$$p:\quad \pm 1,\ \pm 2,\ \pm 3,\ \pm 6.$$

$$q:\quad 1,\ 2.$$

$$\frac{p}{q}:\quad \pm\frac{1}{2},\ \pm 1,\ \pm\frac{3}{2},\ \pm 2,\ \pm 3,\ \pm 6.$$

It is not necessary to list both positive and negative values of q as no further values of $\frac{p}{q}$ would be obtained. The values of $\frac{p}{q}$ are listed in order of increasing magnitude, since it is usually simpler to test them in this order.

The student may show by synthetic division that $\pm\frac{1}{2}$ and ± 1 are not roots of the equation. For our trial of $+\frac{3}{2}$, we have the division

$$\begin{array}{rrrr|l} 2 & +1 & -2 & -6 & \frac{3}{2} \\ & +3 & +6 & +6 & \\ \hline 2 & +4 & +4 & 0 & \end{array}$$

The zero remainder shows that $\frac{3}{2}$ is a root and $x - \frac{3}{2}$ is a factor of the left side of the equation. The second factor is $2x^2 + 4x + 4$, and the

remaining roots may be found by writing

$$2x^2 + 4x + 4 = 0,$$

or

$$x^2 + 2x + 2 = 0.$$

The quadratic formula applied to this equation gives $x = -1 \pm i$. Hence, the three roots of the given equation are $\frac{3}{2}$, $-1 \pm i$. *Ans.*

If a factor corresponding to a root is removed, as in the preceding example, an equation of degree 1 less than that of the original equation is the result. This new equation is called a **depressed** equation with respect to the given equation.

The theorem of this section also enables us to find the real rational factors of a polynomial $P(x)$ with integral coefficients. For instance, Example 1 shows that

$$\begin{aligned} 2x^3 + x^2 - 2x - 6 &= (x - \tfrac{3}{2})(2x^2 + 4x + 4) \\ &= (2x - 3)(x^2 + 2x + 2). \end{aligned}$$

If the coefficients of an equation are rational fractions, both members may be multiplied by the LCD of the fractions to obtain an equivalent equation with integral coefficients.

EXAMPLE 2. Show that the equation $x^4 + 3x^2 + 7x + 6 = 0$ has no rational roots.

Solution: Since $a_0 = 1$, the only possible rational roots are the integral factors of the constant term 6. Furthermore, if any positive number is substituted for x, each term of the left member of the equation will be a positive number, and a sum of positive numbers is never zero. Therefore, the possibilities for rational roots are -1, -2, -3, and -6. We may show by synthetic division that none of these numbers is a root and, hence, that the equation has no rational roots.

The argument whereby positive roots were excluded as possibilities for the equation in Example 2 applies in general. *A rational integral equation with all its coefficients positive can have no positive root.*

NOTE: By inspection of an imaginary number such as $-1 + i$, the student may see that the words "positive" and "negative" can be applied only to real numbers. Consequently, he should understand that any positive or negative number is automatically restricted to be real.

An equation whose roots are respectively the negatives of the roots of a given equation $P(x) = 0$ may be formed by replacing x by $-x$; for if a given number is a zero of $P(x)$, the negative of this number is a

zero of $P(-x)$. The effect of replacing x by $-x$ is to change the signs of the odd powers of x in the polynomial. Thus, if

$$P(x) = 2x^3 + x^2 - 2x - 6,$$

$$P(-x) = -2x^3 + x^2 + 2x - 6.$$

We found in Example 1 that $2x^3 + x^2 - 2x - 6 = 0$ has the roots $\frac{3}{2}$, $-1 + i$, and $-1 - i$, and it may easily be verified that the equation $-2x^3 + x^2 + 2x - 6 = 0$ has the roots $-\frac{3}{2}$, $1 - i$, and $1 + i$.

It follows now that *a rational integral equation with no missing powers and with its coefficients alternating in sign can have no negative root.* For, if $P(x)$ has alternating signs and no missing powers, $P(-x)$ has coefficients all of one sign. The student may complete the argument.

Illustration: The equation $x^3 - 5x^2 + 2x - 7 = 0$ has no negative roots; for all the powers of x from the third down are present, and the coefficients alternate in sign.

EXERCISES 90

Find all the roots of each of the following equations:

1. $x^3 - 4x^2 + x + 6 = 0$
2. $y^3 + y^2 - 22y - 40 = 0$
3. $v^3 + 3v^2 - 5v - 39 = 0$
4. $z^3 - 6z^2 + 13z - 10 = 0$
5. $3x^3 + 19x^2 + 14x - 90 = 0$
6. $2w^3 - 15w^2 + 38w - 30 = 0$
7. $4u^3 - 11u^2 + u + 1 = 0$
8. $y^4 - 4y^3 - 5y^2 + 36y - 36 = 0$
9. $x^4 - 6x^3 - x^2 + 34x + 8 = 0$
10. $3v^4 - 2v^3 + 2v^2 + 10v + 3 = 0$
11. $12z^4 + 5z^3 + 10z^2 + 5z - 2 = 0$
12. $24x^4 - 8x^3 - 44x^2 + 7x + 12 = 0$
13. $6y^4 - 13y^3 + 2y^2 - 4y + 15 = 0$
14. $4w^5 + 4w^4 - 5w^3 + 25w^2 - 84w + 36 = 0$
15. $20x^5 - 9x^4 - 74x^3 + 30x^2 + 42x - 9 = 0$

112. Imaginary Roots

In many of the advanced applications of algebra, it is necessary to have information about the imaginary roots of rational integral equations with real coefficients. One of the basic theorems is considered in this section.

Theorem: *If a rational integral equation* $P(x) = 0$ *with real coefficients has an imaginary root* $a + bi$ $(b \neq 0)$, *then it also has the root* $a - bi$; *that is, imaginary roots occur only in conjugate pairs.*

In order to prove this theorem, we need an auxiliary result, namely, if

$$P(a + bi) = C + Di,$$

where C and D are real, then,

$$P(a - bi) = C - Di.$$

This result follows for any polynomial $P(x)$ if we can show that it holds in the special case where $P(x) = x^m$, for a polynomial with real coefficients is a sum of terms of the type ax^m. We shall prove the law for the special case by the use of mathematical induction.

If $m = 1$, so that $P(x) = x$,

$$P(a + bi) = a + bi,$$

and

$$P(a - bi) = a - bi.$$

Now, assume that for any integral value of m, say $m = s$ [which means that $P(x) = x^s$] we have

$$(a + bi)^s = C + Di,$$

and

$$(a - bi)^s = C - Di.$$

Next, multiply both sides of the first equation by $a + bi$, and both sides of the second equation by $a - bi$ to find

$$\begin{aligned}(a + bi)^{s+1} &= aC - bD + (aD + bC)i \\ &= C_1 + D_1 i,\end{aligned}$$

and

$$\begin{aligned}(a - bi)^{s+1} &= aC - bD - (aD + bC)i \\ &= C_1 - D_1 i,\end{aligned}$$

where the values of C_1 and D_1 are the same in both cases.

This discussion shows that if the required result is correct for any integral value of m, it is also correct for the succeeding integral value. Since the result has been verified for $m = 1$, it follows for $m = 2, 3, 4,$ and so on. Thus, the induction is complete.

We may now prove the original theorem as follows:

Let $$P(a + bi) = C + Di \qquad (C \text{ and } D \text{ real}).$$

Then, $$P(a - bi) = C - Di.$$

Since $a + bi$ is assumed to be a root of $P(x) = 0$, we have

$$C + Di = 0.$$

But a complex number is zero if, and only if, both the real and imaginary parts are zero. Hence,

$$C = 0 \qquad \text{and} \qquad D = 0.$$

Consequently, $$P(a - bi) = 0,$$

and $a - bi$ is also a root of $P(x) = 0$.

Illustration: In Example 1 of the preceding section, we found that the equation $2x^3 + x^2 - 2x - 6 = 0$ has the conjugate imaginary roots $-1 + i$ and $-1 - i$.

Example 1. If the equation $x^4 + 2x^3 + 2x^2 + 10x + 25 = 0$ has the root $1 - 2i$, what are the other roots?

Solution: If $1 - 2i$ is a root, $1 + 2i$ must be a second root, and a factor of the left member of the equation is

$$(x - 1 + 2i)(x - 1 - 2i),$$

or $$x^2 - 2x + 5.$$

The other factor of the left side of the equation may be found by ordinary long division to be $x^2 + 4x + 5$. Thus, the depressed equation still to be solved is

$$x^2 + 4x + 5 = 0.$$

For this equation, the quadratic formula gives $x = -2 \pm i$. Hence, the required roots are $1 + 2i$, $-2 + i$, and $-2 - i$. *Ans.*

Example 2. Form an equation of lowest possible degree with real coefficients which has $\sqrt{2}$ and $3 - i$ for two of its roots.

Solution: Since the equation is to have real coefficients and $3 - i$ as an imaginary root, we must have $3 + i$ as another root. Therefore, the required equation is

$$(x - \sqrt{2})(x - 3 + i)(x - 3 - i) = 0,$$

or $$x^3 - (6 + \sqrt{2})x^2 + (10 + 6\sqrt{2})x - 10\sqrt{2} = 0. \quad \textit{Ans.}$$

Notice that the coefficients in the answer to Example 2 are real but not rational. The student may show that a fourth-degree equation with real, rational coefficients may be obtained by assuming a fourth

root $x = -\sqrt{2}$ in addition to the three previously used. The situation illustrated by these remarks is covered by the following theorem:

Theorem: *If a rational integral equation with rational coefficients has a root* $a + \sqrt{b}$, *where* a *and* b *are rational numbers and* b *is not a perfect square, then the equation has* $a - \sqrt{b}$ *as a second root.*

The proof of this statement is similar to the proof of the preceding theorem and is left for the student.

EXAMPLE 3. If $-2 + \sqrt{5}$ is a root of the equation

$$x^4 + 4x^3 + 4x^2 + 20x - 5 = 0,$$

find the remaining roots.

Solution: Since the coefficients are rational and $-2 + \sqrt{5}$ is a root, $-2 - \sqrt{5}$ must be a second root. Consequently,

$$(x + 2 - \sqrt{5})(x + 2 + \sqrt{5}),$$

or

$$x^2 + 4x - 1$$

is a factor of the left member of the equation. The other factor is found by division to be $x^2 + 5$. So the depressed equation yet to be solved is

$$x^2 + 5 = 0;$$

and we find $x = \pm i\sqrt{5}$. Hence, the required roots are $-2 - \sqrt{5}$, $i\sqrt{5}$, and $-i\sqrt{5}$. *Ans.*

EXERCISES 91

In each of Exercises 1 to 6, form the equation of lowest degree with real coefficients if two of its roots are the given numbers. Let 1 be the coefficient of the term of highest degree in each case.

1. $-5,\ 2 + 3i$ **2.** $4,\ -2 + i$ **3.** $-2i,\ 1 - 3i$

4. $i\sqrt{5},\ -i\sqrt{2}$ **5.** $\sqrt{6},\ 1 - 2i$ **6.** $\sqrt{2},\ 3 + 4i$

In each of Exercises 7 to 12, form the equation of lowest degree with real, rational coefficients if two of its roots are the given numbers. Let 1 be the coefficient of the term of highest degree in each case.

7. $5,\ 2 - \sqrt{5}$ **8.** $-3,\ 1 - 2\sqrt{2}$ **9.** $3 - \sqrt{2},\ \sqrt{10}$

10. $\sqrt{7},\ -1 + 3\sqrt{2}$ **11.** $2 - i,\ 1 + \sqrt{3}$ **12.** $4 + i,\ 2 - \sqrt{6}$

13. One root of $3x^4 - 16x^3 + 24x^2 + 44x - 39 = 0$ is $3 + 2i$. Find the other roots.

14. One root of $5u^4 + 34u^3 + 40u^2 - 78u + 51 = 0$ is $-4 - i$. Find the other roots.

15. Find the remaining roots of $2y^4 - 22y^3 + 65y^2 - 24y - 3 = 0$ if one root is $3 - \sqrt{6}$.

16. Find the remaining roots of $4v^4 - 46v^3 + 135v^2 - 138v + 54 = 0$ if one root is $5 + \sqrt{7}$.

17. In each of the following, let 1 be the coefficient of the term of highest degree:

(*a*) Form the equation of lowest degree which has the roots $3i$ and $\sqrt{5}$.

(*b*) Form the equation of lowest degree with real coefficients if two of its roots are $3i$ and $\sqrt{5}$.

(*c*) Form the equation of lowest degree with real, rational coefficients if two of its roots are $3i$ and $\sqrt{5}$.

18. In each of the following, let 1 be the coefficient of the term of highest degree:

(*a*) Form the equation of lowest degree which has the roots $-2i$ and $2 - \sqrt{3}$.

(*b*) Form the equation of lowest degree with real coefficients if two of its roots are $-2i$ and $2 - \sqrt{3}$.

(*c*) Form the equation of lowest degree with real, rational coefficients if two of its roots are $-2i$ and $2 - \sqrt{3}$.

113. Multiplication of the Roots by a Constant

It is often convenient to be able to write an equation whose roots are multiples, respectively, of the roots of a given equation. Let the given equation be

$$P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0, \tag{1}$$

and suppose we wish to find an equation each of whose roots is m times the corresponding root of Equation (1). We may let $y = mx$ or $x = \dfrac{y}{m}$ and substitute this value for x in Equation (1) to obtain

$$a_0\left(\frac{y}{m}\right)^n + a_1\left(\frac{y}{m}\right)^{n-1} + \cdots + a_{n-1}\left(\frac{y}{m}\right) + a_n = 0.$$

This equation may be simplified by multiplying both sides by m^n to get

$$a_0y^n + a_1my^{n-1} + \cdots + a_{n-1}m^{n-1}y + a_nm^n = 0. \tag{2}$$

Since for each value of x which satisfies Equation (1), there is a

corresponding value of y, where $y = mx$, which satisfies Equation (2), we have the required result. In applying this result, missing powers of x must be replaced by zeros.

EXAMPLE 1. Write an equation whose roots are the roots of the following equation each multiplied by 2: $x^3 + 3x^2 - x - 3 = 0$.

Solution: We have $m = 2$ and $y = 2x$. Hence, after substituting $x = y/2$ and multiplying the two members of the equation by 2^3, we find the required equation is

$$y^3 + 3(2)y^2 - (2^2)y - 3(2^3) = 0$$

or

$$y^3 + 6y^2 - 4y - 24 = 0. \quad \textit{Ans.}$$

The student may verify the fact that the roots of the given equation are -3, -1, and 1, and those of the final equation are -6, -2, and 2.

EXAMPLE 2. Write an equation whose roots are the roots of the following equation each multiplied by $\frac{1}{10}$:

$$2x^4 + 50x^3 + 1000x - 10{,}000 = 0.$$

Solution: The required result may be obtained by writing $10y$ in place of x to give

$$2(10^4)y^4 + 50(10^3)y^3 + 1000(10)y - 10{,}000 = 0,$$

$$2(10^4)y^4 + 5(10^4)y^3 + (10^4)y - (10^4) = 0,$$

or

$$2y^4 + 5y^3 + y - 1 = 0. \quad \textit{Ans.}$$

The procedure used in Example 2 is to be preferred to the method of substituting for m in Equation (2) because of the possibility that we may forget to supply zeros for the missing powers. Many mathematicians prefer to use this method in every case.

Although we may write an equation whose roots are respectively the negatives of the roots of a given equation by using the value $m = -1$ in Equation (2), it is simpler to replace x by $-x$ in the given equation. This procedure is evidently the equivalent of the first method and has the advantage that the result may be written at once by changing the signs of the odd powers of x.

EXAMPLE 3. Write the equation whose roots are, respectively, the negatives of the roots of the equation

$$P(x) = x^5 + 3x^4 - 6x^3 + x^2 - 1 = 0.$$

Solution: The required equation is

$$P(-x) = -x^5 + 3x^4 + 6x^3 + x^2 - 1 = 0,$$

or

$$x^5 - 3x^4 - 6x^3 - x^2 + 1 = 0. \quad Ans.$$

EXERCISES 92

In each of Equations 1 to 12, transform the given equation so that its roots are each multiplied by the number following the semicolon.

1. $x^2 + x - 6 = 0;\ 5$
2. $v^2 - 2v - 8 = 0;\ 3$
3. $z^3 + 8z - 7 = 0;\ 4$
4. $2y^4 - 3y - 11 = 0;\ 2$
5. $4u^4 - 2u^3 + u - 6 = 0;\ -2$
6. $5x^4 - x^3 + x - 1 = 0;\ -4$
7. $y^6 - 4y^5 + y^4 - 2y^3 + 17 = 0;\ -1$
8. $2x^7 - x^6 + 3x^4 - x^2 + 6x + 1 = 0;\ -1$
9. $7w^3 - 3w^2 - 18w + 54 = 0;\ \frac{1}{3}$
10. $s^4 - 16s^2 - 16s - 24 = 0;\ \frac{1}{2}$
11. $4u^3 + 7u^2 - 13 = 0;\ 10$
12. $v^3 + 8v^2 - 6 = 0;\ \frac{1}{10}$

For each of the following examples, transform the equation so that each root is multiplied by 10^k. Then choose a positive or negative integral value for k so that the resulting coefficients are the smallest integers possible. (NOTE: Equations of the type given here occur, among other places, in electric circuit analysis.)

13. $x^3 + 710x^2 + 4500x + 68{,}000 = 0$
14. $11y^4 + 120{,}000y^2 + 8{,}600{,}000 = 0$
15. $5t^4 - 0.3t^3 + 0.007t + 0.0006 = 0$
16. $u^4 + (1.75)(10^{-2})u^3 + (1.85)(10^{-6})u^2 + 10^{-13} = 0$

114. Descartes's Rule of Signs

A polynomial $P(x)$, with real coefficients and arranged in descending powers of x, is said to have a **variation in sign** if two consecutive terms have opposite signs.

Illustration: The polynomial $x^5 - 4x^4 - 2x^3 + x - 1$ has three variations in sign: one from x^5 to $-4x^4$; one from $-2x^3$ to $+x$; and one from $+x$ to -1. Note that some powers of x may be missing.

If a sequence of plus and minus signs is written, starting and ending with the same sign, there will always be an even number of variations of sign or there will be none; whereas if opposite signs begin and end the sequence, there will be an odd number of variations in sign. These statements become obvious if we think of crossing from one side to the

other of a straight line; for if we are to start and end on the same side, the line must be crossed an even number of times or not at all; if we are to start and end on opposite sides, an odd number of crossings must be made.

We can now show that a polynomial with real coefficients and with only negative zeros and imaginary zeros has an even number of variations in sign or none. There will be no loss in generality if we take 1 as the coefficient of the highest power of x; for this coefficient may be factored out without affecting the number of variations in sign. Corresponding to a negative zero $-a$, where a is positive, the polynomial must have a factor $x + a$. Furthermore, corresponding to imaginary zeros, which must occur in conjugate pairs, there will be factors of the type $x^2 \pm bx + c$, where $b^2 - 4c$ is negative and, hence, where c is positive. If any number of factors of the two specified types are multiplied together, we shall obtain a polynomial whose term of highest degree has $+1$ for a coefficient and whose constant term is positive. By referring to the discussion of the preceding paragraph, we see that the polynomial must have an even number of variations in sign or no variations at all.

From the last result, there follows immediately an important corollary, namely, *if a rational integral equation with real coefficients has an odd number of variations in sign, it must have at least one positive root.*

We suppose next that $P(x)$ has a factor $x - r$, where r is positive, and we compare the number of variations of sign in $P(x)$ with the number of variations of sign in the quotient $Q(x)$ obtained when $P(x)$ is divided by $x - r$. Let the pattern of signs in $Q(x)$ be represented by

$$Q(x):\ + \cdots + - \cdots - + \cdots,$$

where the dots after a sign are to mean that more signs of the same kind may occur but no change in sign takes place until it is explicitly indicated.

Since $P(x) = (x - r)Q(x)$, we may obtain the pattern of signs in $P(x)$ by the following schematic multiplication:

$$\begin{array}{rcccccccccccc}
xQ(x): & + & & \cdots & + & - & & \cdots & - & + & & \cdots & \\
-rQ(x): & & - & \cdots & - & + & & \cdots & + & - & & \cdots & - \\
\hline
(x-r)Q(x): & + & ? & \cdots & ? & - & ? & \cdots & ? & + & ? & \cdots\ ? & -
\end{array}$$

The question marks in the third line represent ambiguous signs; these would be known only if the specific coefficients in the multiplication were known. However, no matter what the doubtful signs are, it is evident that at least one variation, and in general an odd number of

variations, occurs as we go from any definite sign to the next definite sign. Thus, there is at least one variation in $P(x)$ corresponding to each variation in $Q(x)$, and $P(x)$ has at least one additional variation at the end. Hence, the *smallest* number of variations that $P(x)$ can have is one more than $Q(x)$ has.

It has been noted that in passing from any definite sign to the next definite sign in $P(x)$ there may be an odd number of variations of sign. Therefore, there may be an even number of variations in addition to the one that must occur. Consequently, $P(x)$ may have an even number of variations more than the minimum number described in the preceding paragraph. Thus, the number of variations in $P(x)$ may exceed the number of variations in $Q(x)$ by 1 or by 1 plus an even number—in other words, by an odd number.

Now, let $P(x) = 0$ have k positive roots, $r_1, r_2, \cdots, r_k$. Then, we have

$$\begin{aligned} P(x) &= (x - r_1)Q_1(x), \\ Q_1(x) &= (x - r_2)Q_2(x), \\ &\cdots\cdots\cdots\cdots \\ Q_{k-1}(x) &= (x - r_k)Q_k(x), \end{aligned}$$

where $Q_1(x)$ has an odd number, say $2m_1 + 1$, of variations in sign less than $P(x)$, $Q_2(x)$ has an odd number, say $2m_2 + 1$, of variations less than $Q_1(x)$, and so on. If $P(x)$ has no more than k positive zeros, the polynomial $Q_k(x)$ either is a constant or has only negative zeros and imaginary zeros. As we have previously seen in this case, $Q_k(x)$ has an even number of variations or none. Let $2m$ stand for the number of variations in $Q_k(x)$. Then, if $Q_k(x)$ has $2m_k + 1$ fewer variations in sign than $Q_{k-1}(x)$, we have, for the total number of variations in sign of $P(x)$:

$$2m + 2m_k + 1 + \cdots + 2m_2 + 1 + 2m_1 + 1,$$

or

$$2(m + m_k + \cdots + m_2 + m_1) + k.$$

Since the m's may be zeros or positive integers,

$$2(m + m_k + \cdots + m_2 + m_1)$$

is zero or an even integer. From this result, we have

Descartes's Rule of Signs: *The number of positive roots of a rational integral equation* P(x) = 0 *with real coefficients is either equal to the number of variations in sign of* P(x) *or is less than that number by an even integer.*

Information concerning the negative roots may be obtained by applying the rule of signs to the equation $P(-x) = 0$. This possibility follows from the fact that the positive roots of this equation are the negative roots of $P(x) = 0$.

EXAMPLE 1. Apply Descartes's rule of signs to the equation

$$x^4 - 3x^2 + 2x - 1 = 0.$$

Solution: Since the polynomial $P(x)$ has three variations in sign, the equation $P(x) = 0$ has three positive roots or one positive root. Also,

$$P(-x) = x^4 - 3x^2 - 2x - 1$$

has one variation in sign, so $P(x) = 0$ has one negative root. This information may be displayed as in the table to the right. In any row of the table showing a possible grouping of the roots, the total number of roots is four, the degree of the equation; in the column for imaginary roots, the number is zero or an even integer since imaginary roots must occur in conjugate pairs.

+	−	i
3	1	0
1	1	2

EXAMPLE 2. Apply Descartes's rule of signs to $x^7 - 2x^5 - x^2 = 0$, and state, in tabular form, conclusions as to the permissible types of roots.

Solution: We first remove the factor x^2 which corresponds to the fact that $x = 0$ is a double root of the equation. We next apply the rule of signs to the depressed equation $x^5 - 2x^3 - 1 = 0$, whose left member has one variation in sign. Also, $P(-x) = -x^5 + 2x^3 - 1$ has two variations in sign. Since the depressed equation is of the fifth degree, there are five nonzero roots in all. Thus, the given equation has two zero roots, one positive root, and either two or no negative roots. As the table shows, there may be two or four imaginary roots.

+	−	i
1	2	2
1	0	4

EXERCISES 93

Apply Descartes's rule of signs to each of the following equations, and state, in tabular form, conclusions as to the permissible types of roots.

1. $y^3 + 3y^2 + 4 = 0$

2. $w^3 - 5w^2 - 1 = 0$

3. $4x^3 - x - 5 = 0$

4. $u^4 + 7u^3 + 2u^2 - 10 = 0$

5. $3z^4 - 2z^2 + 6z - 8 = 0$

6. $5v^4 - 3v^2 + 6 = 0$

7. $5x^4 - 3x^2 - 6 = 0$

8. $y^6 - 2y^3 + y - 7 = 0$

9. $u^6 - 2u^5 + 3u^3 + 2 = 0$

10. $r^6 - 4r^2 + 9 = 0$

11. $y^7 + 4y^6 - y^5 + y^2 - 2y - 3 = 0$
12. $w^8 - 5w^6 + w^3 - 6w^2 - w + 4 = 0$
13. $x^9 + 2x^6 + x^5 + 5x^4 + 3x^3 + x^2 - 5 = 0$
14. $v^{2k+1} - b^{2k+1} = 0$; (k integral, $b > 0$)
15. $y^{2k} - a^{2k} = 0$; (k integral)
16. Show that the equation $P(x) = 0$ has no real roots if all the terms are of even degree and all the coefficients are positive.
17. Use Descartes's rule of signs to show that the equation $P(x) = 0$ has no negative roots if the coefficients are alternating in sign and there are no missing powers of x.

115. Location of Roots

We have previously seen that the real roots of an equation $P(x) = 0$ are given by the abscissas of the points where the graph of $y = P(x)$ meets the X axis. By reading the coordinates of these points as accurately as possible from the graph, the real roots of the equation may be located at least approximately.

Example 1. Locate the roots of the equation $x^3 - 4x^2 - 3x + 15 = 0$ between consecutive integers.

Solution: We make a table of values and a graph of the function $y = x^3 - 4x^2 - 3x + 15$ (see Figure 73). The value of y corresponding to each chosen value of x may be determined by the use of synthetic division.

x	y
-2	-3
-1	$+13^*$
0	$+15$
1	$+9$
2	$+1$
3	-3^*
4	$+3^*$

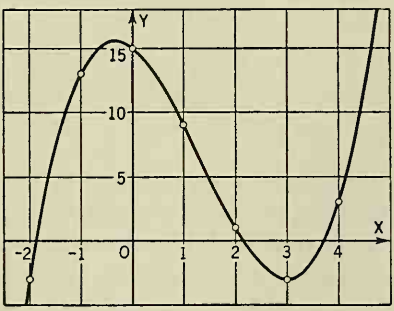


Fig. 73

From the graph we read that there is a root between -2 and -1; another between 2 and 3; and a third between 3 and 4. *Ans.*

The real roots of an equation can frequently be located without drawing the graph by using only the fact that the curve is an unbroken line. It follows that:

The curve must cross the X *axis at least once between any two values of* x *for which the corresponding values of* y *are opposite in sign.*

The starred places in the table for Example 1 indicate where the sign of y changes.

It is helpful to be able to limit the range of values of x to be considered when we attempt to locate the real zeros of a polynomial. In this connection, we have the following simple result:

If the "indicated" divisor in a synthetic division is positive and the signs in the third line of the division are all positive, then no real number greater than the divisor can be a zero of the dividend.

This statement follows from the fact that if any greater divisor were used, the numbers in the third line would all be increased and there could not be a zero remainder. If there are zeros in the third line, the statement is still valid provided no variations in sign occur. (It is assumed that the term of highest degree has a positive coefficient.) We may apply this result to the equation $P(-x) = 0$ to obtain corresponding information about the negative roots of $P(x) = 0$; that is, if K is an upper bound for the positive roots of $P(-x) = 0$, $-K$ will be a lower bound for the negative roots of $P(x) = 0$.

Example 2. Use the preceding discussion to find two integers between which are located all the real roots of the equation

$$x^3 + 5x^2 - 6x + 3 = 0.$$

Solution: The synthetic division of the given polynomial by $x - 1$ is as follows:

$$\begin{array}{rrrr|r} 1 & +5 & -6 & +3 & 1 \\ & +1 & +6 & 0 & \\ \hline 1 & +6 & 0 & +3 & \end{array}$$

The fact that there is no variation in sign in the third line shows that the equation has no real root greater than 1. Note that we pay no attention to the zero in the third line in counting the variations in sign. Also, 1 itself is not a root because the remainder is not zero.

By changing x to $-x$, we obtain the equation

$$-x^3 + 5x^2 + 6x + 3 = 0,$$

or

$$x^3 - 5x^2 - 6x - 3 = 0.$$

It may be verified that 7 is the smallest positive integral value which may be used as a divisor to give no minus signs in the third line of the synthetic division, where $x^3 - 5x^2 - 6x - 3$ is the dividend. There-

fore, all the real roots of the given equation lie between -7 and $+1$. *Ans.*

EXERCISES 94

In each of the following exercises, use the foregoing method to find upper and lower bounds for the roots, that is, two integers between which are located all the real roots of the given equation. Then determine two consecutive integers between which each real root is located.

1. $2x^3 + x - 11 = 0$ **2.** $y^3 + 3y - 21 = 0$
3. $w^3 + 2w + 40 = 0$ **4.** $v^3 + 4v^2 + 9v + 9 = 0$
5. $x^3 - 3x^2 + 1 = 0$ **6.** $z^3 - 7z + 5 = 0$
7. $y^3 - 5y^2 - 46y - 36 = 0$ **8.** $v^4 - 11v - 50 = 0$
9. $z^4 - 20z^2 - 21z - 22 = 0$ **10.** $x^4 - 9x^3 + 15x^2 + 14x + 23 = 0$

116. The Method of Successive Approximations

In the successive-approximation method for finding the real roots of an equation $f(x) = 0$, we first locate the desired root between two points on the graph of $y = f(x)$. If the curve is replaced by a straight line, that is, the chord joining the two points, we find that the abscissa of the point where the chord cuts the X axis furnishes an approximation to the desired root which may be closer than the x coordinate of either of the two points (see Figure 74). This new approximation may be used to obtain two new points which are closer together than the first two points and which have the root between them. This idea may be used to find the root correct to any desired number of decimal places as in the next example.

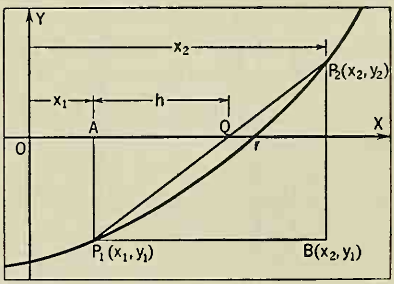


Fig. 74

Referring to Figure 74, let $AQ = h$. This distance is to be added to x_1 to reach the point where the line P_1P_2 cuts the X axis. From the similar triangles AP_1Q and BP_2P_1 we have

$$\frac{h}{P_1A} = \frac{P_1B}{BP_2},$$

or

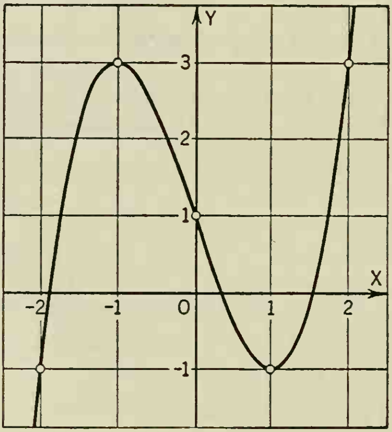
$$h = P_1A \cdot \frac{P_1B}{BP_2}.$$

If we agree to proceed from left to right, then P_1A is always the numerical value of y_1, P_1B is $x_2 - x_1$, and BP_2 is the sum of the numerical values of y_1 and y_2. Hence,

$$h = \frac{|y_1|}{|y_1| + |y_2|}(x_2 - x_1).$$

EXAMPLE 1. Find, correct to three decimal places, the larger positive root of $x^3 - 3x + 1 = 0$.

Solution: The table of values and the graph for $y = x^3 - 3x + 1$ (Figure 75) show that the given equation has three real roots: one between -2 and -1; one between 0 and 1; and the third between 1 and 2.



x	y
-2	-1
-1	$+3$
0	$+1$
1	-1
2	$+3$

Fig. 75

The largest root which is between 1 and 2 appears from the graph to be at 1.5, approximately. Calculating the value of y for $x = 1.5$, we find $y = -0.125$. The minus sign shows that the curve lies below the X axis at this point. Since $y = +3$ when $x = 2$, the desired root is greater than 1.5. Furthermore, for $x = 1.6$ the value of y is $+0.296$, which shows that the root lies between 1.5 and 1.6. We calculate h, the "correction" to be added to the value 1.5, as follows:

$$\left.\begin{array}{l} x_1 = 1.5,\ y_1 = -0.125 \\ x_2 = 1.6,\ y_2 = +0.296 \end{array}\right\} \quad h = \frac{0.125}{0.125 + 0.296}(0.1) = 0.03-.$$

We now have $1.5 + 0.03 = 1.53$ as a new approximate value of the root. Calculating the corresponding value of y, we find $y = -0.0084$,

approximately. Also, for $x = 1.54$, $y = +0.0323$, approximately. Consequently, the root is between 1.53 and 1.54. The second correction is now calculated.

$$\left.\begin{aligned} x_1 &= 1.53,\ y_1 = -0.0084 \\ x_2 &= 1.54,\ y_2 = +0.0323 \end{aligned}\right\} \quad h = \frac{0.0084}{0.0084 + 0.0323}\,(0.01) = 0.002+.$$

Hence, a third approximation to the root is 1.532. For this value of x, we find $y = -0.00036$, and for $x = 1.533$, $y = +0.00369$. Using the last two points, the correction is less than 0.0001. Hence, the required root is 1.532, correct to three decimal places. *Ans.*

We may also use this method to find the other two roots of the equation to be 0.347 and -1.879, each correct to three decimal places. Alternatively, we may use the approximate depressed equation $x^2 + 1.532x - 0.6530 = 0$, which gives the same values when solved by the quadratic formula. This quadratic equation is obtained by dividing the given polynomial, $x^3 - 3x + 1$, by $x - 1.532$, the factor corresponding to the approximate root 1.532, and discarding the remainder.

The scheme of depressing the equation has the advantage of yielding an equation of lower degree to be solved for the remaining roots. However, there is also the disadvantage that the depressed equation is not exact so that the accuracy with which its roots approximate the corresponding roots of the original equation will not be obvious.

In the case of real roots, we can always determine the accuracy of an approximate root by direct substitution. Thus, to check the approximate root 0.347, we substitute 0.3475 and 0.3465 into the original polynomial and find the values -0.0002 and $+0.0021$, respectively. Since these last two are opposite in sign, we know that the root is greater than 0.3465 and less than 0.3475. Hence, we may guarantee that 0.347 is actually correct to three decimal places. Notice that the magnitude of the two remainders is not involved in the argument; only the fact that they are opposite in sign is of importance.

The method of successive approximation is not limited in its applicability to rational integral equations. It may also be used for solving equations which involve other types of functions; this fact is illustrated by the next example.

Example 2. Find, correct to three decimal places, the real root of the equation $x + \log x - 2 = 0$.

Solution: If the given equation is written in the form $\log x = 2 - x$, we see that the roots will be given by the abscissas of the points of

intersection of the graphs of $y = \log x$ and $y = 2 - x$. Figure 76 shows that these two graphs intersect at a single point whose x coordinate is approximately 1.7.

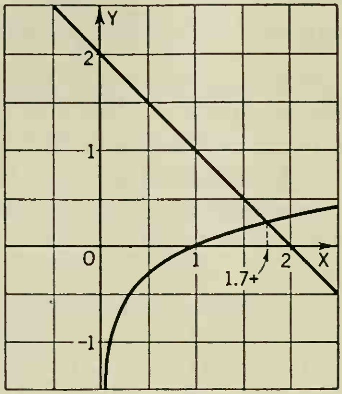


Fig. 76

Using Table II (Appendix) we make the calculations shown in the following schedule, wherein $y = x + \log x - 2$:

x	y	
1.7 1.8	-0.06955 $+0.05527$	$h_1 = \dfrac{0.06955}{0.12482}(0.1) = 0.05+$
1.75 1.76	-0.00696 $+0.00551$	$h_2 = \dfrac{0.00696}{0.01247}(0.01) = 0.005+$
1.755 1.756	-0.00072 $+0.00052$	$h_3 = \dfrac{0.00072}{0.00124}(0.001) = 0.0005+$
1.7555 1.7556	-0.00010 $+0.00002$	

Since the desired root lies between 1.7555 and 1.7556, we have 1.756 as the required result correct to three decimal places. *Ans.*

EXERCISES 95

In Equations 1 to 6, find the indicated root correct to two decimal places.

1. $x^3 + 6x - 23 = 0$; the positive root
2. $y^3 + 2y^2 + 15 = 0$; the negative root

3. $x + \log x - 4 = 0$; the real root
4. $3v + \log v - 7 = 0$; the real root
5. $\sqrt{x} + \sqrt[3]{x} = 1$; the real root
6. $\sqrt[3]{x} + 2x = 5$; the real root

In Numbers 7 to 22 find the indicated roots correct to three decimal places

7. $z^3 + 2z + 47 = 0$; the negative root
8. $x^3 + 3x^2 - 10 = 0$; the positive root
9. $y^3 + 68 = 0$; the real root
10. $x^3 - 37 = 0$; the real root
11. $2x^3 - 9x^2 - 17x - 10 = 0$; the positive root
12. $5v^3 - 8v^2 - 11v + 15 = 0$; the negative root
13. $y^3 - 6y^2 - y + 23 = 0$; all the roots
14. $x^3 + 5x^2 - 28x - 34 = 0$; all the roots
15. $2x^4 - 5x^3 - 22x + 9 = 0$; all the roots
16. $3y^4 + 11y^3 + 6y + 13 = 0$; all the roots
17. $2y - 6 \log y = 3$; the two real roots
18. $4z - 31 \log z = 0$; the two real roots

NOTE: Other equations may be taken from Exercises 97.

19. A rectangular solid has a volume of 43 cu ft. Find the dimensions if they form an arithmetic progression with a common difference of 2.

20. The volume of a tank is to be quadrupled by increasing each dimension by the same amount. If the tank is initially 2 by 5 by 7 ft, what must be the amount of the increase in each dimension?

21. A cube has a volume equal to that of a sphere whose radius is 2 in. shorter than an edge of the cube. What is the length of the radius of the sphere?

22. An open box is formed by cutting equal squares from each corner of a rectangular piece of tin 8 by 14 in. and then folding up the sides. If the volume of the box thus formed is 52 cu in., what is the length of the edge of the cut-out squares?

117. Diminishing or Increasing the Roots by a Constant

Another method for obtaining an approximation to a real irrational root of a rational integral equation depends on the process of diminishing or increasing the roots of the equation by a constant. We shall see in this section how to form an equation whose roots are those of a given equation, each diminished by an amount h.

Let the given equation be

$$a_0x^n + a_1x^{n-1} + \cdots + a_n = 0. \tag{1}$$

If in this equation we replace x by $x_1 + h$, so that $x_1 = x - h$, we have

$$a_0(x_1 + h)^n + a_1(x_1 + h)^{n-1} + \cdots + a_n = 0. \tag{2}$$

Imagine the terms of Equation (2) expanded by the binomial theorem; and let the like powers of x_1 be collected to give the result

$$a_0x_1^n + A_1x_1^{n-1} + \cdots + A_{n-2}x_1^2 + A_{n-1}x_1 + A_n = 0, \tag{3}$$

where the A's stand for new coefficients, at present unknown. Since $x_1 = x - h$, the roots of Equation (3) will be the roots of Equation (1), each diminished by h.

In order to find the unknown coefficients, the A's, let us regain Equation (1) from Equation (3) by replacing x_1 by $x - h$. The result is

$$\begin{aligned} a_0(x - h)^n &+ A_1(x - h)^{n-1} + \cdots \\ &+ A_{n-2}(x - h)^2 + A_{n-1}(x - h) + A_n = 0, \end{aligned} \tag{4}$$

which would reduce to Equation (1) if the terms were expanded and like powers of x collected. However, important information can be obtained from Equation (4) as it stands. If this equation is divided termwise by $x - h$, the quotient is

$$a_0(x - h)^{n-1} + A_1(x - h)^{n-2} + \cdots + A_{n-2}(x - h) + A_{n-1} = Q_1(x),$$

and the remainder is A_n. Therefore, if the left side of Equation (1), which is Equation (4) in disguise, is divided by $x - h$, the remainder is the constant term of the desired equation, that is, Equation (3).

Furthermore, if $Q_1(x)$ is divided by $x - h$, we find that the quotient is

$$a_0(x - h)^{n-2} + A_1(x - h)^{n-3} + \cdots + A_{n-2} = Q_2(x),$$

and the remainder is A_{n-1}, another coefficient of the desired equation. Clearly, this successive process of division may be continued until all the A's are found. In the next example, the continued division is carried out by a convenient synthetic process.

Example 1. Form the equation whose roots are the roots of $x^3 + 3x^2 - x - 3 = 0$, each diminished by 2.

Solution:

$$\begin{array}{rrrr|l}
1 & +3 & -1 & -3 & 2 \\
 & +2 & +10 & +18 & \\
\hline
1 & +5 & +9 & \mathbf{+15} & \\
 & +2 & +14 & & \\
\hline
1 & +7 & \mathbf{+23} & & \\
 & +2 & & & \\
\hline
\mathbf{1} & \mathbf{+9} & & &
\end{array}$$

The successive remainders 15, 23, 9, and 1 are the required coefficients in order from the constant term to the third-degree term. Thus, the new equation is

$$x_1^3 + 9x_1^2 + 23x_1 + 15 = 0. \quad \textit{Ans.}$$

Notice that we have divided the given polynomial synthetically by $x - 2$, the first quotient again by $x - 2$, and so on. It may be verified that the given equation has the roots -3, -1, and 1 and that the equation just obtained has the roots -5, -3, and -1.

EXAMPLE 2. Form the equation whose roots are the roots of the equation given in Example 1 each increased by 0.1.

Solution: Since increasing the roots by 0.1 is equivalent to diminishing the roots by -0.1, we may use the same scheme as in the preceding example. The coefficients of $x^3 + 3x^2 - x - 3$ are written as shown in the next schedule to aid in keeping the decimal point in proper position.

1	+3.0	−1.00	−3.000	−0.1
	−0.1	−0.29	+0.129	
1	+2.9	−1.29	**−2.871**	
	−0.1	−0.28		
1	+2.8	**−1.57**		
	−0.1			
1	**+2.7**			

Thus, the required equation is

$$x_1^3 + 2.7x_1^2 - 1.57x_1 - 2.871 = 0. \quad \textit{Ans.}$$

It is instructive to make a graphical interpretation of the transformation of this section. For this purpose, consider Figure 77 which shows a curve whose equation may be assumed to be in the form $y = f(x)$ and two vertical axes labeled Y and Y_1. The Y_1 axis has been taken h units to the right of the Y axis. It is clear from the figure that if P is a point on the curve whose distance from the Y axis is x, and whose distance from the Y_1 axis is x_1, we have $x = x_1 + h$ or $x_1 = x - h$. Hence, if we wish to refer the curve to the Y_1 axis, in other

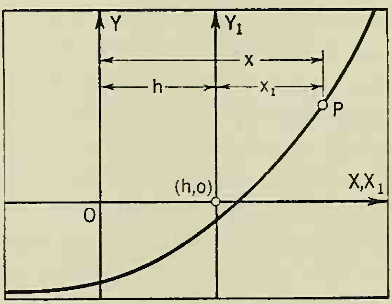


Fig. 77

words, to measure horizontal distances from the Y_1 axis, the equation to be used is $y_1 = f(x_1 + h)$. Thus, the effect of replacing x by $x_1 + h$ in the equation $y = f(x)$ is to shift the origin h units to the right, or $|h|$ units to the left if h is negative. If $f(x) = 0$ has a root $x = r$, then $f(x_1 + h) = 0$ has a corresponding root $x_1 = r_1$ which is h units less than r.

EXERCISES 96

In each of Examples 1 to 10, obtain an equation for which each root is a root of the given equation diminished by the number which follows the semicolon.

1. $y^2 + 2y - 8 = 0;\ \ 4$

2. $w^2 + w - 42 = 0;\ \ 5$

3. $x^3 + 10x - 11 = 0;\ \ 3$

4. $3u^3 - 2u^2 - 4 = 0;\ \ -2$

5. $2z^4 - z^2 - 9 = 0;\ \ 5$

6. $x^4 - 6x^2 - 16x + 21 = 0;\ \ 3$

7. $v^4 + v - 2 = 0;\ \ -1$

8. $2y^4 - y^3 - 1 = 0;\ \ -4$

9. $4s^3 - 3s^2 + 20 = 0;\ \ 0.3$

10. $5v^3 - 8v^2 - 8 = 0;\ \ -0.2$

For each of the following equations, make a graph of the polynomial on the left side. Then diminish the roots of the equation by the amount indicated after the semicolon, and show the translated vertical axis on your graph.

11. $x^2 - 4x - 6 = 0;\ \ 5$

12. $x^2 + 8x + 2 = 0;\ \ -7$

13. $x^3 + 15x - 23 = 0;\ \ 2$

14. $x^3 - 2x^2 - 7 = 0;\ \ 3$

118. Horner's Method

Horner's method for the approximation of a real, irrational root of a rational, integral equation makes use of the material in the preceding section in a rather interesting fashion. Suppose that we can locate a real root of an equation between two consecutive integers. Then, we may form an equation whose roots are those of the original equation each diminished by one of these integers. The corresponding root of the second equation must be numerically between 0 and 1. Suppose that we locate it more closely between two successive tenths. Then, we may form a third equation with roots diminished by one of these tenths. The third equation will have its corresponding root numerically between 0 and 0.1. Clearly this process may be carried to a point where the corresponding root of the transformed equation is numerically as small as we please. If the process is carried to n steps, the successive x's will be related by the equations

$$x = x_1 + h_1,\ x_1 = x_2 + h_2,\ \cdots,\ x_{n-1} = x_n + h_n,$$

where h_k is the amount by which the roots are diminished during the kth transformation. Suppose, further, that at this stage the corresponding root of the transformed equation is small enough to be negligible for the purpose at hand. Then putting $x_n = 0$, we have by successive substitution,

$$\begin{aligned} x &= x_1 + h_1 \\ &= x_2 + h_2 + h_1 \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &= x_{n-1} + h_{n-1} + \cdots + h_2 + h_1 \\ &= h_n + h_{n-1} + \cdots + h_2 + h_1 \end{aligned}$$

as an approximation to the desired root. This result means that the approximation is obtained as the sum of the successive amounts by which the roots of the given equation are diminished.

EXAMPLE 1. Show that the equation $x^3 - 2x^2 - 8 = 0$ has only one real root and find its value correct to three decimal places.

Solution: By Descartes's rule of signs, the equation has one positive root. The equation that results if x is replaced by $-x$ has no variations

x	y
0	-8
1	-9
2	-8
3	$+1$
4	$+24$

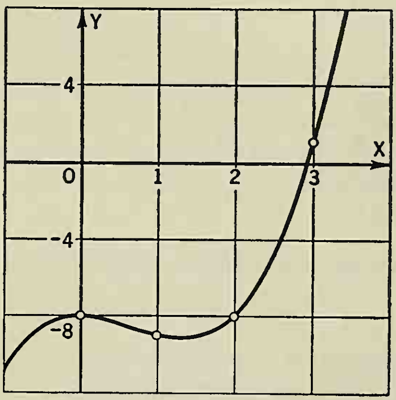


Fig. 78

in sign; hence, the given equation has no negative roots. We see, therefore, that the original equation has only one real root.

In order to locate the one real root, we make a table of values and plot the graph of $y = x^3 - 2x^2 - 8$ (see Figure 78). From the table,

we see that the required root is between 2 and 3, so we diminish the roots of the equation by 2.

1	-2	0	-8	2	$x_1 = x + 2$
	$+2$	0	0		
1	0	0	**-8**		
	$+2$	$+4$			
1	$+2$	**$+4$**			
	$+2$				
1	**$+4$**				

The transformed equation is

$$x_1^3 + 4x_1^2 + 4x_1 - 8 = 0, \tag{1}$$

and the required root of this equation is between 0 and 1. From the graph, we estimate that the original root is between 2.9 and 3, which would mean that the corresponding root of Equation (1) is between 0.9 and 1. This estimate is verified by substituting $x_1 = 0.9$ and $x_1 = 1$ into the left member of Equation (1), which gives the values -0.431 and 1, respectively. Accordingly, we diminish the roots again, this time by 0.9.

1	$+4.0$	$+\ 4.00$	-8.000	$+0.9$	$x_1 = x_2 + 0.9$
	$+0.9$	$+\ 4.41$	-7.569		
1	$+4.9$	$+\ 8.41$	**-0.431**		
	$+0.9$	$+\ 5.22$			
1	$+5.8$	**$+13.63$**			
	$+0.9$				
1	**$+6.7$**				

The second transformed equation is

$$x_2^3 + 6.7x_2^2 + 13.63x_2 - 0.431 = 0, \tag{2}$$

and this equation has a root between 0 and 0.1 corresponding to the required root of the given equation.

For values of x_2 between 0 and 0.1, the values of the x_2^2 and x_2^3 terms of Equation (2) will be small in comparison with the first-degree term. Consequently, a good approximation to the next digit of the required root may be obtained by using only the linear part of the equation, that is,

$$13.63x_2 - 0.431 = 0.$$

This equation gives $x = 0.03$, approximately. By synthetic division we find that the desired root is between 0.03 and 0.04. Therefore, we diminish the roots again by 0.03.

$$
\begin{array}{lllll|l}
1 & +6.70 & +13.6300 & -0.431000 & & 0.03 \quad x_2 = x_3 + 0.03 \\
  & +0.03 & +0.2019 & +0.414957 & & \\
\hline
1 & +6.73 & +13.8319 & \mathbf{-0.016043} & & \\
  & +0.03 & +0.2028 & & & \\
\hline
1 & +6.76 & \mathbf{+14.0347} & & & \\
  & +0.03 & & & & \\
\hline
\mathbf{1} & \mathbf{+6.79} & & & &
\end{array}
$$

Consequently, the third transformed equation is

$$x_3^3 + 6.79x_3^2 + 14.0347x_3 - 0.016043 = 0,$$

and it has a root between 0 and 0.01 corresponding to the root of the given equation. We use the linear part of Equation (3) to find $x_3 = 0.001$, approximately, for the next digit of the desired root. We carry out one line of the next step as a check.

$$
\begin{array}{llll|l}
1 & +6.790 & +14.0347 & -0.01604 & 0.001 \quad x_4 = x_3 + 0.001 \\
  & +0.001 & +\ 0.0068 & +0.01404 & \\
\hline
1 & +6.791 & +14.0415 & -0.00200 &
\end{array}
$$

Notice that the last two coefficients have been rounded off to four and five decimal places, respectively. If this transformation were completed, it is evident that the linear term would be increased by comparatively little. Therefore, a root of the fourth transformed equation is approximated by solving

$$14.0415x_4 - 0.00200 = 0,$$

which gives

$$x_4 = 0.0001 +.$$

We now add the numbers 2, +0.9, +0.03, and +0.001 by which the roots of the respective equations were diminished; this addition gives $x = 2.931$ as the approximate value, correct to three decimal places, for the desired root.

The computations in the foregoing discussion may be put into a more compact form as follows:

1	-2	0	-8	2	$h_1 = 2$
	$+2$	0	0		
1	0	0	-8		
	$+2$	$+4$			
1	$+2$	$+4$			
	$+2$				
1	$\mathbf{+4.0}$	$\mathbf{+\ 4.00}$	$\mathbf{-8.000}$	0.9	$h_2 = 0.9$
	$+0.9$	$+\ 4.41$	$+7.569$		
1	$+4.9$	$+\ 8.41$	-0.431		
	$+0.9$	$+\ 5.22$			
1	$+5.8$	$+13.63$			
	$+0.9$				
1	$\mathbf{+6.70}$	$\mathbf{+13.6300}$	$\mathbf{-0.431000}$	0.03	$h_3 = 0.03$
	$+0.03$	$+\ 0.2019$	$+0.414957$		
1	$+6.73$	$+13.8319$	-0.016043		
	$+0.03$	$+\ 0.2028$			
1	$+6.76$	$+14.0347$			
	$+0.03$				
1	$\mathbf{+6.790}$	$\mathbf{+14.0347}$	$\mathbf{-0.01604}$	0.001	$h_4 = 0.001$
	$+0.001$	$+\ 0.0068$	$+0.01404$		
1	$+6.791$	$+14.0415$	-0.00200		

$$x = h_1 + h_2 + h_3 + h_4 = 2.931, \text{ approximately. } \quad \textit{Ans.}$$

The following suggestions may be made for approximating a positive root by Horner's method. The student should have clearly in mind the general discussion at the beginning of this section. He may then proceed as follows:

(1) *Obtain all possible information about the roots by Descartes's rule.*

(2) *Find all rational roots and depress the equation to as low a degree as possible. Draw a graph, using the depressed equation, and locate the irrational roots.* This graph may be quite rough except in the vicinity of those points where the curve intersects the X axis.

(3) *Diminish the roots by the largest integer that is less than the desired root.* (If this root is between 0 and 1, this step may be omitted.)

(4) *Locate the desired root of the transformed equation between two consecutive tenths and diminish by the smaller one. A carefully drawn graph is helpful at this stage.*

(5) *Locate the root of the second transformed equation between two consecutive hundredths, and diminish by the smaller one.* (Solving the linear part of this equation frequently gives the next digit of the root. The student may always check, however, by using the usual method of location, that is, find two consecutive hundredths for which the left member of the equation has opposite signs.)

(6) *Proceed in the same fashion for as many steps as are needed to find the root to the desired accuracy. The required approximation is the algebraic sum of the amounts by which the roots of the original equation have been diminished.*

It is to be observed that the foregoing illustration was concerned with the determination of a positive root. If, however, the required root is a negative one, we can change the signs of the roots and use exactly the procedure that was illustrated.

NOTE: Many computers prefer to diminish in each step by the digit that gives the closest approximation rather than to approach the root always from the one side. Thus, in the preceding problem, the closest value for h_1 is 3 rather than 2, and if this value is used, then h_2 comes out -0.1 in place of 0.9. The student may check to see that the remaining steps are precisely as before. This plan has the advantage that no divisor greater than 5 is used for diminishing the roots. Furthermore, at any stage where the two schemes do not diminish by the same digit, the use of the closer digit gives a transformed equation whose linear part usually yields the better approximation to the diminished root. However the h's are not always positive as in the preceding schedule.

Since a cubic equation always has at least one real root, it is possible to find the approximate values of the other roots in the manner indicated in the next example.

EXAMPLE 2. Find the imaginary roots of the equation in Example 1.

Solution: We return to the schedule on page 342, where we found 2.931 for the real root of the given equation and complete the last transformation to find

$$x_4^3 + 6.793x_4^2 + 14.0483x_4 - 0.00200 = 0.$$

(The last two coefficients have been rounded off to the given number of decimal places).

We know from our work that the real root of the equation in x_4 is close to zero. Accordingly, we neglect the constant term and divide

the resulting equation by x_4 to obtain the quadratic equation

$$x_4^2 + 6.793x_4 + 14.0483 = 0.$$

It may be verified that the roots of this equation are

$$x_4 = -3.396 \pm 1.585i, \text{ approximately.}$$

We now add 2.931 to obtain the corresponding roots of the original equation; they are

$$x = -0.465 \pm 1.585i. \quad \textit{Ans.}$$

As a partial verification of the results in the last two examples, we may make the following comparison. A cubic whose roots are r_1, r_2, and r_3 may be written as

$$(x - r_1)(x - r_2)(x - r_3) = 0,$$

or

$$x^3 - (r_1 + r_2 + r_3)x^2 + \cdots - r_1r_2r_3 = 0,$$

from which we see that the negative of the sum of the roots is the coefficient of x^2 and the negative of the product of the roots is the constant term, the coefficient of x^3 being taken as unity. If we add the roots 2.931, $-0.465 + 1.585i$, and $-0.465 - 1.585i$, we obtain 2.001 in place of 2, the negative of the coefficient of x^2 in the original equation. Also, the product of the three roots is 7.996 in place of 8, the negative of the given constant term.

EXERCISES 97

For each of the following equations, find the indicated root or roots, correct to three decimal places:

1. $y^3 + y - 11 = 0$; the positive root
2. $3v^3 + v + 1 = 0$; the negative root
3. $2x^3 - 9x^2 + 7x + 4 = 0$; the negative root
4. $3y^3 - 5y^2 - 24y - 19 = 0$; the positive root
5. $z^3 - 4z^2 - 4z - 7 = 0$; all the roots
6. $u^3 + 4u - 17 = 0$; all the roots
7. $w^4 + 4w^3 - 37 = 0$; the real roots
8. $4x^4 + 9x^3 + 3 = 0$; the real roots

NOTE: Other problems may be taken from Exercises 95.

119. Relations between Roots and Coefficients

Let the roots of a rational integral equation of the nth degree with

$a_0 = 1$ be given: $r_1, r_2, \cdots, r_n$. Then the equation must be

$$(x - r_1)(x - r_2) \cdots (x - r_n) = 0,$$

or

$$x^n - (r_1 + r_2 + \cdots + r_n)x^{n-1} + (r_1r_2 + r_1r_3 + \cdots)x^{n-2} - \cdots$$
$$+ (-1)^n r_1r_2 \cdots r_n = 0.$$

Thus, if we make the leading coefficient unity and write

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_n = 0,$$

we see that

$-p_1 = r_1 + r_2 + \cdots + r_n$, *the sum of the roots;*

$p_2 = r_1r_2 + r_1r_3 + \cdots$, *the sum of the products of the roots, two at a time;*

$-p_3 = r_1r_2r_3 + \cdots$, *the sum of the products of the roots, three at a time:*

. . . .

$(-1)^n p_n = r_1r_2 \cdots r_n$, *the product of all the roots.*

For example, if the equation is

$$2x^4 - 3x^3 + 6x + 4 = 0,$$

we may divide both members by 2, and write

$$x^4 - \tfrac{3}{2}x^3 + 0x^2 + 3x + 2 = 0.$$

Therefore

$$r_1 + r_2 + r_3 + r_4 = \tfrac{3}{2};$$

$$r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4 = 0;$$

$$r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4 = -3;$$

$$r_1r_2r_3r_4 = 2.$$

EXERCISES 98

In Problems 1 to 9 the numbers given are roots of an equation. Form equations from the given roots by making use of the relations between the roots and the coefficients.

1. $-5, 1, 6$ **2.** $-4, -3, 5$ **3.** $2, 3 \pm 2i$

4. $-2, -1, 2, 3$ **5.** $-3, 1, 2, 4$ **6.** $2, 3, 4, 5$

7. $1, 4, 2 \pm \sqrt{3}$ **8.** $5, 6, 1 \pm i$ **9.** $1 \pm \sqrt{2}, 2 \pm i$

In Equations 10 to 15 determine the values of all coefficients which are represented by a or b. Find all the roots of each equation.

10. $2x^3 - 3x^2 - 18x + b = 0;\quad r_1 + r_2 = 4$

11. $y^3 - 5y^2 - 16y + b = 0;\quad r_1 = -r_2$

12. $v^3 - av^2 + av - 3 = 0;\quad r_1 = 1/r_2$

13. $z^4 - 8z^3 + 12z^2 + az + b = 0;\quad r_1 = r_2,\ r_3 = r_4$

14. $x^4 + 12x^3 + 44x^2 + ax + b = 0;\quad r_1 = r_2,\ r_3 = r_4$

15. $y^3 - 3y^2 - 24y + a = 0;$ roots are in arithmetic progression

16. Show that one root of $x^3 + bx^2 + cx + d = 0$ is the negative of the other if $bc = $ d.

17. If r_1, r_2, r_3 are the roots of $v^3 + bv^2 + cv + d = 0$, evaluate $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$, $(d \neq 0)$.

18. Form the equation whose roots are the squares of the roots of $u^2 - bu + c = 0$.

19. Show that $b^2 - 2c$ is the sum of the squares of the roots of the equation $x^3 + bx^2 + cx + d = 0$.

20. If the equation $w^3 + 3aw + 2b = 0$ has a double root, show that $a^3 + b^2 = 0$.

21. Show that $a^3c = b^3$, if $x^3 + ax^2 + bx + c = 0$ has a triple root.

120. The General Cubic Equation

By dividing both members by the coefficient of x^3, we may write the general cubic equation in the form

$$x^3 + p_1x^2 + p_2x + p_3 = 0. \tag{1}$$

Since the sum of the roots is $-p_1$, an increase of $\frac{p_1}{3}$ in each of the three roots would result in an equation with the second-degree term missing. Consequently, if we make the substitution

$$x + \frac{p_1}{3} = y \quad \text{or} \quad x = y - \frac{p_1}{3}, \tag{2}$$

we shall arrive at an equation of the form

$$y^3 + Py + Q = 0, \tag{3}$$

where P and Q will be given in terms of the original coefficients. We shall take (3) as the standard form of the cubic equation.

In attempting to solve Equation (3), we gain some freedom by replacing the one unknown y by the sum of two unknowns, say

$$y = s + t; \tag{4}$$

$s + t$ is to be a root of Equation (3), but, beyond that fact, we are still free to impose somewhat arbitrarily another condition on s and t before their determination.

In order to see what this arbitrary condition might be in order to facilitate the solution of the equation, let us make the substitution for y. We get

$$s^3 + 3s^2t + 3st^2 + t^3 + P(s + t) + Q = 0,$$

or

$$s^3 + t^3 + (3st + P)(s + t) + Q = 0. \tag{5}$$

An obvious simplification is now at hand by choosing, as our second condition on s and t,

$$3st + P = 0,$$

or

$$st = -\frac{P}{3}. \tag{6}$$

This condition reduces Equation (5) to the simpler form

$$s^3 + t^3 = -Q, \tag{7}$$

and, by cubing both members of (6), we have also

$$s^3t^3 = -\frac{P^3}{27}. \tag{8}$$

Since both the sum and the product of s^3 and t^3 are given by the last two equations, it follows that s^3 and t^3 are the roots of the quadratic equation

$$u^2 + Qu - \frac{P^3}{27} = 0. \tag{9}$$

(Compare Section 119.)

Since the roots of Equation (9) are

$$-\frac{Q}{2} \pm \sqrt{\frac{Q^2}{4} + \frac{P^3}{27}},$$

we may take

$$s^3 = -\frac{Q}{2} + \sqrt{\frac{Q^2}{4} + \frac{P^3}{27}}, \tag{10}$$

and

$$t^3 = -\frac{Q}{2} - \sqrt{\frac{Q^2}{4} + \frac{P^3}{27}}. \tag{11}$$

Equations (10) and (11) give three values for s and three values for t, respectively. However, these values must be so paired that Equation (6), $st = -P/3$, is satisfied.

Let the imaginary cube roots of unity, that is, the imaginary roots of $x^3 - 1 = 0$, be denoted by ω and ω^2, so that

$$\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2},$$

$$\omega^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

Then, if s_1 and t_1 are such that $s_1t_1 = -P/3$, the only other possible pairs of values for s and t are $s = \omega s_1$, $t = \omega^2 t_1$ and $s = \omega^2 s_1$, $t = \omega t_1$.

Thus the three roots of Equation (3) are

$$s_1 + t_1, \ \omega s_1 + \omega^2 t_1, \text{ and } \omega^2 s_1 + \omega t_1.$$

These solutions of the cubic are usually called Cardan's formulas after Cardan who first published them in 1545. However, Cardan apparently obtained them from Tartaglia under a false promise of secrecy. In any case, we know now that the solution of the cubic is probably due originally to del Ferro about 1505. A detailed account of this interesting story will be found in *A History of Mathematics* by Florian Cajori (2nd Edition, The Macmillan Company, 1919, pages 133–134).

EXAMPLE 1. Solve the equation $x^3 + 3x^2 + 9x + 5 = 0$.

Solution: Since $p_1 = 3$, we have $p_1/3 = 1$. Accordingly, we first substitute $x = y - 1$ and find

$$y^3 + 6y - 2 = 0.$$

Now let $y = s + t$ so that

$$s^3 + t^3 + (3st + 6)(s + t) - 2 = 0.$$

Next put $3st + 6 = 0$ or

$$st = -2.$$

Then

$$s^3 + t^3 = 2,$$

and

$$s^3t^3 = -8.$$

Therefore the quadratic equation whose roots are s^3 and t^3 is

$$u^2 - 2u - 8 = 0.$$

The roots of this equation are easily seen to be

$$u = 4 \quad \text{and} \quad u = -2.$$

Take

$$s^3 = 4 \quad \text{and} \quad t^3 = -2.$$

If principal roots are used, the values

$$s_1 = \sqrt[3]{4} \quad \text{and} \quad t_1 = -\sqrt[3]{2}$$

satisfy the condition $s_1t_1 = -2$. Consequently the three roots of $y^3 + 6y - 2 = 0$ are

$$y_1 = \sqrt[3]{4} - \sqrt[3]{2},$$

$$y_2 = \omega\sqrt[3]{4} - \omega^2\sqrt[3]{2},$$

$$= -\frac{1}{2}(\sqrt[3]{4} - \sqrt[3]{2}) + i\frac{\sqrt{3}}{2}(\sqrt[3]{4} + \sqrt[3]{2}),$$

and

$$y_3 = \omega^2\sqrt[3]{4} - \omega\sqrt[3]{2}$$

$$= -\frac{1}{2}(\sqrt[3]{4} - \sqrt[3]{2}) - i\frac{\sqrt{3}}{2}(\sqrt[3]{4} + \sqrt[3]{2}).$$

The check is left for the student. The corresponding roots of the original equation may now be written

$$x_1 = \sqrt[3]{4} - \sqrt[3]{2} - 1,$$

$$x_2 = -\left(1 + \frac{\sqrt[3]{4} - \sqrt[3]{2}}{2}\right) + i\frac{\sqrt{3}}{2}(\sqrt[3]{4} + \sqrt[3]{2}),$$

$$x_3 = -\left(1 + \frac{\sqrt[3]{4} - \sqrt[3]{2}}{2}\right) - i\frac{\sqrt{3}}{2}(\sqrt[3]{4} + \sqrt[3]{2}). \quad \textit{Ans.}$$

Frequently the answers are left in terms of ω.

An inspection of Equations (10) and (11) reveals the following information for the cubic with real coefficients:

(1) If $\dfrac{Q^2}{4} + \dfrac{P^3}{27} > 0$, then s^3 and t^3 are both real as in Example 1. If s_1 and t_1 are the real cube roots, then the roots of the cubic equation are

$$s_1 + t_1, \quad \omega s_1 + \omega^2 t_1, \quad \omega^2 s_1 + \omega t_1.$$

Of these roots, the first is real, and the other two are the conjugate imaginary numbers,

$$-\frac{s_1 + t_1}{2} + i\frac{s_1 - t_1}{2}\sqrt{3}$$

and

$$-\frac{s_1 + t_1}{2} - i\frac{s_1 - t_1}{2}\sqrt{3}.$$

This case offers no practical difficulty although the final forms of the roots may present a cumbersome appearance.

(2) If $\frac{Q^2}{4} + \frac{P^3}{27} = 0$, then Equation (3) has real, equal roots, $s^3 = t^3$. Here $s_1 = t_1$ may again be taken as the real cube root. The roots of the cubic equation are now

$$2s_1, \quad (\omega + \omega^2)s_1, \quad (\omega^2 + \omega)s_1,$$

that is,

$$2s_1, \quad -s_1, \quad -s_1.$$

At least two of the roots are equal in this case, and if $s_1 = 0$, the three roots are equal.

(3) If $\frac{Q^2}{4} + \frac{P^3}{27} < 0$, then s^3 and t^3 are both imaginary of the form $a + ib$ and $a - ib$, respectively. Since the product of s and t must be real, s and t must in this case be conjugate imaginary. Let $s_1 = c + id$ and $t_1 = c - id$ be a set of required cube roots. Then the roots of the cubic itself are

$$s_1 + t_1 = 2c,$$

$$\omega s_1 + \omega^2 t_1 = -c - d\sqrt{3},$$

and

$$\omega^2 s_1 + \omega t_1 = -c + d\sqrt{3},$$

which are all real. However, there is a practical difficulty involved in getting the exact values of c and d, since there is no general algebraic method for extracting cube roots of imaginary numbers. The only method usually available is the trigonometric method of Section 98, the use of which is illustrated in the next example.

Example 2. Solve the equation $y^3 - 3y + 1 = 0$.

Solution: Let $y = s + t$. Then

$$s^3 + t^3 + (3st - 3)(s + t) + 1 = 0.$$

Choose
$$3st - 3 = 0,$$
or
$$st = 1.$$
Then
$$s^3 + t^3 = -1,$$
and
$$s^3t^3 = 1.$$

Thus the quadratic equation to be solved is

$$u^2 + u + 1 = 0,$$

which has the roots ω and ω^2. Hence

$$s^3 = \omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2},$$

and
$$t^3 = \omega^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

Now
$$\omega = \cos 120° + i \sin 120°$$
so that the three values of s are

$$s_1 = \cos 40° + i \sin 40°,$$

$$s_2 = \cos 160° + i \sin 160°,$$

and
$$s_3 = \cos 280° + i \sin 280°.$$

Since the corresponding values of t are the respective conjugate imaginary numbers, we may write the roots of the cubic at once:

$$y_1 = s_1 + t_1 = 2 \cos 40°,$$

$$y_2 = s_2 + t_2 = 2 \cos 160°,$$

and
$$y_3 = s_3 + t_3 = 2 \cos 280°. \quad \textit{Ans.}$$

EXERCISES 99

Solve Equations 1 to 12 by the method of this section.

1. $x^3 + 6x + 2 = 0$

2. $y^3 + 60y - 160 = 0$

3. $v^3 - 12v + 20 = 0$

4. $x^3 - 18x - 42 = 0$

5. $z^3 + 9z + 6 = 0$

6. $w^3 + 24w + 16 = 0$

7. $2x^3 + 6x + 3 = 0$

8. $4y^3 + 6y - 1 = 0$

9. $u^3 + 6u^2 + 27u + 18 = 0$
10. $x^3 - 9x^2 + 45x - 75 = 0$
11. $x^3 - 3x - 1 = 0$
12. $y^3 - 9y - 9 = 0$
13. Determine the range of values of k for which $x^3 + kx + 2 = 0$ will have exactly one real root.
14. When will the roots of $v^3 - kv + 16 = 0$ be real and distinct? When will this equation have two equal roots?
15. For what values of k will $x^3 - 3k^2x + 2k^3 = 0$ have two equal roots?
16. Determine k so that $y^3 - 6y^2 + (12 - 3k)y + 12k - 8 = 0$ will have two equal roots.

121. Trigonometric Solution for Three Real Roots

In the third case of the preceding section, the case where all three roots are real, there is a more practical method than the use of Cardan's formulas for finding the roots. It appears from the solution of Example 2 that it might be desirable to represent a solution in the form

$$y = k \cos \theta. \tag{1}$$

Let us substitute this expression for y into the cubic

$$y^3 + Py + Q = 0.$$

The result is
$$k^3 \cos^3 \theta + Pk \cos \theta + Q = 0,$$

or
$$\cos^3 \theta + \frac{P}{k^2} \cos \theta = -\frac{Q}{k^3}.$$

This last equation reminds us of the trigonometric identity

$$4 \cos^3 \theta - 3 \cos \theta = \cos 3\theta,$$

that is,
$$\cos^3 \theta - \tfrac{3}{4} \cos \theta = \tfrac{1}{4} \cos 3\theta.$$

By comparing the coefficients, we have

$$k = \sqrt{-\tfrac{4}{3}P}, \tag{2}$$

which is real (since $\dfrac{Q^2}{4} + \dfrac{P^3}{27} < 0$ and hence P is negative).

In addition

$$\cos 3\theta = -\frac{4Q}{k^3},$$

or
$$\cos 3\theta = -\frac{Q}{2\sqrt{-P^3/27}}. \tag{3}$$

Again, 3θ is real because $\dfrac{Q^2}{4} + \dfrac{P^3}{27} < 0$ so that $-1 < \dfrac{Q}{2\sqrt{-P^3/27}} < 1$.

Hence, we can find 3θ from a table of cosines, and the three values of y are

$$y_1 = k \cos \theta,$$

$$y_2 = k \cos (\theta + 120°),$$

and

$$y_3 = k \cos (\theta + 240°).$$

EXAMPLE 1. Solve the equation $y^3 - 3y + 1 = 0$ (Example 2 of the preceding section) by the method of this section.

Solution: From Equation (2) we have

$$k = \sqrt{-\tfrac{4}{3}P} = \sqrt{4} = 2.$$

Equation (3) gives

$$\cos 3\theta = -\frac{Q}{2\sqrt{-P^3/27}} = -\frac{1}{2}.$$

Consequently, we take

$$3\theta = 120°,$$

$$\theta = 40°,$$

and find

$$y_1 = 2 \cos 40°,$$

$$y_2 = 2 \cos 160°,$$

and

$$y_3 = 2 \cos 280°. \quad \textit{Ans.}$$

The student may show that the value 240° for 3θ leads to the same roots in different order.

EXAMPLE 2. Solve the equation $y^3 - 12y - 2 = 0$.

Solution: We have

$$\frac{Q^2}{4} + \frac{P^3}{27} = 1 - \frac{(12)^3}{27} = 1 - 64 = -63,$$

so that all three roots are real. Then

$$k = \sqrt{(-\tfrac{4}{3})(-12)} = \sqrt{16} = 4,$$

and

$$\cos 3\theta = -\frac{Q}{2\sqrt{-P^3/27}} = \frac{1}{8} = 0.125.$$

From a five-place table of cosines, we find

$$3\theta = 82°49.1',$$

so that

$$\theta = 27°36.4' \quad \text{approximately.}$$

Thus

$$y_1 = 4\cos 27°36.4' = 3.545,$$

$$y_2 = 4\cos 147°36.4' = -3.378,$$

$$y_3 = 4\cos 267°36.4' = -0.167,$$

each correct to three decimal places. *Ans.*

EXERCISES 100

Solve each of the following by the trigonometric method.

1. $x^3 - 9x - 9 = 0$

2. $z^3 - 24z - 32 = 0$

3. $v^3 - 27v + 27 = 0$

4. $y^3 - 36y + 72 = 0$

5. $u^3 + 9u^2 + 15u - 17 = 0$

6. $w^3 + 12w^2 + 39w + 37 = 0$

7. $y^3 - 3y^2 + 3 = 0$

8. $x^3 - 6x^2 + 24 = 0$

9. $z^3 - 15z - 12 = 0$

10. $v^3 - 18v + 21 = 0$

11. $x^3 - 3x^2 - 9x + 16 = 0$

12. $y^3 + 6y^2 + 6y - 2 = 0$

122. The Quartic Equation

We present here the method of L. Ferrari, a pupil of Cardan's, for solving the general fourth degree equation. As in the case of the cubic, this method was first published by Cardan in 1545.

Let the general quartic equation be taken in the form

$$x^4 + 2p_1x^3 + p_2x^2 + 2p_3x + p_4 = 0.$$

The springboard for Ferrari's method consists in adding $(ax + b)^2$ to both sides and then choosing a and b so that the new left member is a perfect square. As we shall see, a correct choice of a and b depends upon the solution of a cubic equation. Upon performing the addition, we get

$$x^4 + 2p_1x^3 + (a^2 + p_2)x^2 + (2ab + 2p_3)x + (b^2 + p_4) = (ax + b)^2.$$

Now suppose that the left member is the perfect square

$$(x^2 + p_1x + k)^2 = x^4 + 2p_1x^3 + (p_1^2 + 2k)x^2 + 2kp_1x + k^2.$$

Then, by equating coefficients, we find

$$a^2 + p_2 = p_1^2 + 2k,$$

$$2ab + 2p_3 = 2kp_1,$$

and

$$b^2 + p_4 = k^2.$$

From the second of these equations it follows that

$$a^2b^2 = (kp_1 - p_3)^2.$$

Also we can solve the first and third equations for a^2 and b^2, respectively, so that we can substitute these values into the product a^2b^2 to obtain

$$(2k + p_1^2 - p_2)(k^2 - p_4) = (kp_1 - p_3)^2.$$

This equation, when simplified, is a cubic in k and always has a real root if the original quartic equation has real coefficients. This real value of k may be used to find a and b, whereupon the quartic is broken down into two quadratics

$$x^2 + p_1x + k = +(ax + b),$$

and

$$x^2 + p_1x + k = -(ax + b).$$

It should be clear that Ferrari's method is not going to be practical except when the cubic in k has a simple rational root.

Example 1. Solve the equation $x^4 + 6x^3 + 12x^2 + 14x + 3 = 0$.

Solution: We add $a^2x^2 + 2abx + b^2x^2 = (ax + b)^2$ to both members:

$$x^4 + 6x^3 + (a^2 + 12)x^2 + (2ab + 14)x + (b^2 + 3) = (ax + b)^2.$$

In order for the left member to be the perfect square

$$(x^2 + 3x + k)^2 = x^2 + 6x^3 + (2k + 9)x^2 + 6kx + k^2,$$

we must have

$$a^2 + 12 = 2k + 9, \qquad \text{or} \qquad a^2 = 2k - 3,$$

$$2ab + 14 = 6k, \qquad \text{or} \qquad ab = 3k - 7,$$

and

$$b^2 + 3 = k^2, \qquad \text{or} \qquad b^2 = k^2 - 3.$$

By eliminating a and b from these equations, we find

$$(2k - 3)(k^2 - 3) = (3k - 7)^2,$$

or

$$k^3 - 6k^2 + 18k - 20 = 0.$$

It is easy to verify that this cubic equation has the rational root $k = 2$. Hence, we have

$$a^2 = 2k - 3 = 1.$$

Let us choose $a = 1$. Then, since

$$ab = 3k - 7 = -1,$$

we must have $b = -1$.

Our quartic now assumes the form

$$(x^2 + 3x + 2)^2 = (x - 1)^2.$$

Hence, we have to solve

$$x^2 + 3x + 2 = \pm (x - 1),$$

that is,
$$x^2 + 2x + 3 = 0,$$

and
$$x^2 + 4x + 1 = 0.$$

Thus, the roots of the quartic are $-1 \pm i\sqrt{2}$ and $-2 \pm \sqrt{3}$. *Ans.*

EXERCISES 101

Solve the following equations by Ferrari's method:

1. $x^4 - 16x - 12 = 0$
2. $y^4 + 56y + 15 = 0$
3. $v^4 - 10v^3 - 30v - 9 = 0$
4. $z^4 - 8z^3 + 48z - 36 = 0$
5. $y^4 + 2y^3 - 6y^2 + 14y - 3 = 0$
6. $x^4 - 4x^3 - 2x^2 - 8x - 8 = 0$
7. $u^4 + 6u^3 + 4u^2 - 4u - 12 = 0$
8. $v^4 - 6v^3 - 2v^2 - 24v - 24 = 0$
9. $x^4 - 12x^3 + 38x^2 - 32x + 8 = 0$
10. $y^4 - 14y^3 + 44y^2 - 22y + 3 = 0$
11. $v^4 + 4v^3 + 2v^2 + 4v + 40 = 0$
12. $x^4 + 4x^3 + 5x^2 + 8x + 21 = 0$

123. General Remarks

In this chapter, we have discussed certain theorems and methods that are useful in finding real roots of rational integral equations with numerical coefficients. Frequently, only the real roots are needed, and Horner's method or the method of successive approximations may be employed.

There are, however, certain important applications where it is necessary to find all the roots, the imaginary as well as the real ones. For this purpose, there is a method, called "Graeffe's (root-squaring) method," that determines the approximate values of all the roots in one schedule of operations. Although the fundamental ideas in this

method are simple, their application requires lengthy explanations; and, if any considerable degree of accuracy is required, laborious calculations make the use of a computing machine almost imperative. For further information, reference may be made to the first of the two books listed at the end of these remarks.

We have also seen that it is possible to solve the cubic and the quartic equations in a perfectly general, though not always practical, fashion. The student should not be misled into thinking that such methods always exist. In fact, it has been proved that there can be no algebraic formulas for the roots in terms of the coefficients for the general equation of degree higher than four.

References

Conkwright, N. B. *Introduction to the Theory of Equations*, Ginn and Company, Boston, 1941. (Chapter XII is on Graeffe's method.)

Dickson, L. E. *First Course in Theory of Equations*, John Wiley & Sons, Inc., New York, 1922.

Chapter 18

SYSTEMS INVOLVING QUADRATIC EQUATIONS

124. Systems of Quadratic Equations in Two Variables

The equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \qquad (1)$$

where the coefficients A, B, C, D, E, and F are constants, is the most general equation of the second degree in the two variables x and y. It is shown in analytic geometry that the equation which represents the curve of intersection of a right-circular cone and a plane can always be put in the form given by (1). Conversely, if not all three of the constants A, B, and C are zero, (1) may always be regarded as the equation of a curve of intersection of a plane and a cone. Consequently, a curve of this type is called a **conic section.**

As in the case of two linear equations in two unknowns, we may consider a system of two quadratic equations, each of the type above. However, an attempt to find solutions of such a system will, in general, lead to the problem of solving an equation of the fourth degree in one of the unknowns.

125. Graphs of the Simple Quadratics

Some knowledge of graphs of the simpler forms of the quadratic equation in two variables furnishes an important aid to the work of this chapter.

(*a*) The equation

$$x^2 + y^2 = r^2 \qquad (1)$$

has for its graph a **circle** of radius r with its center at the origin. This

information may be obtained by applying the Pythagorean theorem to the right triangle OMP in Figure 79.

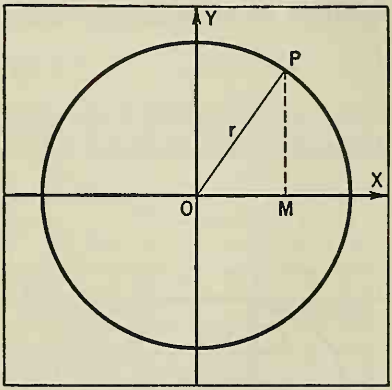


Fig. 79

Illustration: The graph of the equation $x^2 + y^2 = 100$ is a circle of radius 10 units with its center at the origin.

(*b*) The equation

$$ax^2 + by^2 = c, \qquad (2)$$

where a, b, and c are *positive* constants, has an **ellipse** for its graph. It is important to notice that x and y appear only as squares in this equation. This means that if (x_1, y_1) is a point on the curve, then $(-x_1, y_1)$, $(x_1, -y_1)$, and $(-x_1, -y_1)$ are also on the curve. Hence, we say that the curve is *symmetric* with respect to the X axis and to the Y axis. The intercepts on the X axis may be found by putting $y = 0$ to obtain $x = \pm\sqrt{c/a}$; the intercepts on the Y axis are similarly found to be $y = \pm\sqrt{c/b}$.

Illustration: The ellipse whose equation is

$$9x^2 + 25y^2 = 225$$

is shown in Figure 80. The coordinates of points on the curve may be found by first solving the equation for y to obtain

$$y = \pm\tfrac{3}{5}\sqrt{25 - x^2}.$$

It is evident that the largest numerical value for y, which is 3, occurs for the value $x = 0$; also $y = 0$ for $x = 5$ or -5. Furthermore, values of x greater than 5 yield imaginary values of y; we are not at present interested in such values, of course, since we are concerned only with graphs of real values. For the values $x = \pm 3$, we have

$$y = \pm\tfrac{3}{5}\sqrt{25 - 9} = \pm\tfrac{12}{5} = \pm 2.4.$$

Hence, $(3, 2.4)$, $(3, -2.4)$, $(-3, 2.4)$, and $(-3, -2.4)$ are points on the curve. As many additional points as are desired may similarly be found.

(*c*) The equation

$$y = ax^2 + bx + c \qquad (3)$$

has previously been seen to represent a **parabola** with a vertical line of symmetry (see pages 148 to 150). The interchange of x and y in Equation (3) gives an equation of the form

$$x = ay^2 + by + c, \tag{4}$$

which also represents a parabola, this time with a horizontal axis.

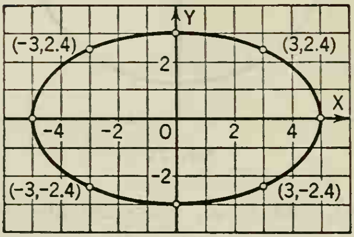


Fig. 80

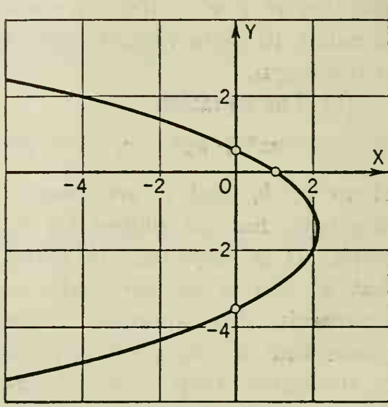


Fig. 81

Illustration: Figure 81 shows the graph of the equation

$$x = -\tfrac{1}{2}y^2 - \tfrac{3}{2}y + 1.$$

Coordinates of points on the parabola may be obtained by substituting values of y and calculating the corresponding values of x.

(*d*) The equation

$$ax^2 - by^2 = c, \tag{5}$$

where a, b, and c are positive constants, has for its graph the curve called a **hyperbola**.

Illustration: The graph of the equation

$$16x^2 - 9y^2 = 144$$

is shown in Figure 82. If the equation is solved for y, there is obtained

$$y = \pm\tfrac{4}{3}\sqrt{x^2 - 9}.$$

Points on the graph may be found in the usual manner. The formula for y shows that $y = 0$ when $x = \pm 3$ and that y is imaginary if x is numerically less than 3. This means that the curve is divided into two separate pieces, called *branches*, one to the left of the point $(-3, 0)$

and the other to the right of the point (3, 0). As in the case of the ellipse, this curve is also symmetric with respect to the coordinate axes.

If x and y are interchanged in Equation (5), there results

$$ay^2 - bx^2 = c, \tag{6}$$

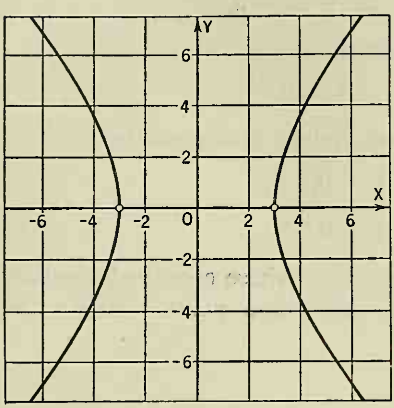


Fig. 82

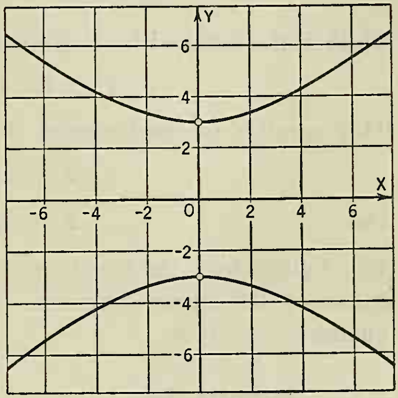


Fig. 83

wherein a, b, and c are positive constants. The graph of Equation (6) is the hyperbola given by Equation (5), but turned through an angle of 90°. This statement is illustrated in Figure 83 which shows the graph of

$$16y^2 - 9x^2 = 144.$$

(*e*) Another form of the equation of the hyperbola which is of practical importance is

$$xy = c, \qquad c \neq 0. \tag{7}$$

Boyle's law, $PV = k$, is an important example of this form. In Figure 84, there appears the graph of the equation

$$xy = 6.$$

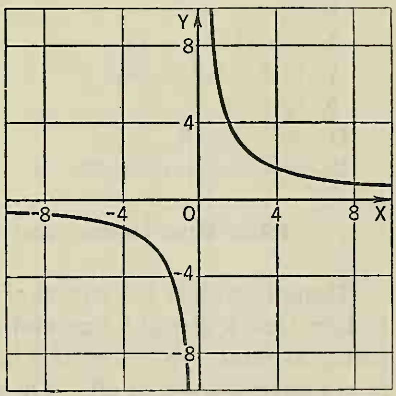


Fig. 84

It is apparent directly from this equation that zero is not a permissible value for either x or y; hence, the hyperbola does not cross either axis.

It is shown in analytic geometry that, if a quadratic equation is satisfied by more than one pair of real values of x and y, then the graph

is either a conic section of the kind previously described (although it might be oriented differently with respect to the coordinate axes) or it is a pair of straight lines, called a *degenerate* conic.

As an example of a degenerate conic, consider the equation

$$(x + 2y - 1)(x - 2y + 1) = 0,$$

which is the factored form of the quadratic

$$x^2 - 4y^2 + 4y - 1 = 0.$$

This equation has for its graph the two straight lines given by

$$x + 2y - 1 = 0,$$

and

$$x - 2y + 1 = 0.$$

This follows from the fact that all the points whose coordinates satisfy either of the linear equations—and only those points—satisfy the quadratic equation.

EXERCISES 102

Identify and sketch each of the curves given by the following equations. In each sketch, label the intercepts of the curve on the coordinate axes.

1. $y^2 = 16x$

2. $4y^2 = x^2$

3. $x^2 + y^2 = 25$

4. $9x^2 + 2y = 0$

5. $y = x^2 - x - 12$

6. $9x^2 + 16y^2 = 144$

7. $16x^2 + 9y^2 = 144$

8. $x = 2y^2 + 3y - 35$

9. $xy = 12$

10. $4y^2 - 9x^2 = 36$

11. $x^2 - y^2 = 9$

12. $xy + 24 = 0$

13. $x^2 - 4xy + 4y^2 = 25$

14. $y^3 - 16x^2y = 0$

126. One Linear and One Quadratic Equation

It appears from the nature of the graphs discussed in the preceding section that a straight line and a conic section may intersect in two points at most. We may also have a straight line tangent to a conic or not intersecting it at all. These three cases are characterized analytically by the three types of solutions that may occur in solving a system composed of a quadratic equation and a linear equation. The system of equations may have two real, distinct solutions; one real solution; or two imaginary solutions.

The analytic solution of such a system may always be obtained as follows:

(1) Solve the linear equation for one of the unknowns.

(2) Substitute the expression which results from step (1) into the quadratic, and simplify to obtain a quadratic equation in one unknown.

(3) Solve the resulting quadratic in one unknown and substitute the roots into the linear equation to find the corresponding values of the second unknown.

(4) The required solutions consist of the corresponding values of the two unknowns *properly paired.*

EXAMPLE 1. Solve both graphically and analytically

$$x^2 + y^2 = 25,$$

$$4x - 3y = 7.$$

Analytic Solution: Solve the linear equation for x to obtain

$$x = \frac{3y + 7}{4}.$$

Substitute this expression for x into the quadratic and find

$$\left(\frac{3y + 7}{4}\right)^2 + y^2 = 25,$$

or

$$25y^2 + 42y - 351 = 0,$$

an equation whose roots are $y = 3$ and $y = -\frac{117}{25}$.

If 3 is substituted for y into the equation $x = \frac{1}{4}(3y + 7)$, we find $x = 4$. Similarly, corresponding to $y = -\frac{117}{25}$, we find $x = -\frac{44}{25}$. Hence, the required solutions are (4, 3) and $(-\frac{44}{25}, -\frac{117}{25})$. *Ans.*

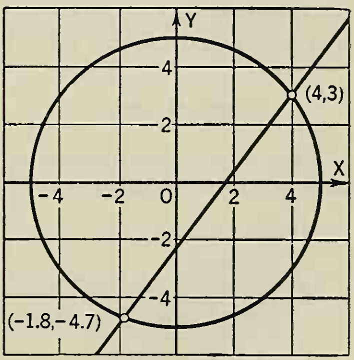


Fig. 85

Graphical Solution: The graphs of the two given equations are shown in Figure 85. The first of these equations is that of a circle of radius 5 units with its center at the origin. This graph may be drawn with compasses. The second equation is linear and may be drawn after plotting two points. The solutions as read from the graph are (4, 3) and $(-1.8, -4.7)$. *Ans.*

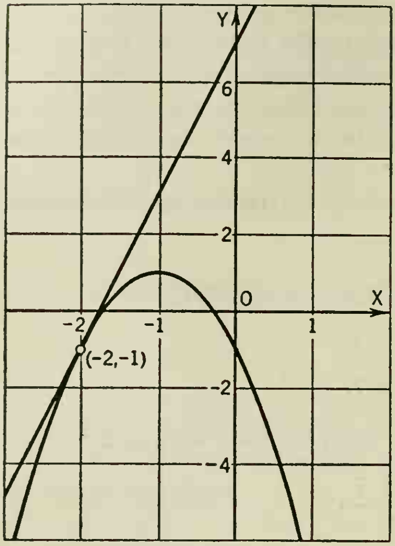


Fig. 86

EXAMPLE 2. Solve both analytically and graphically

$$2x^2 + 4x + y + 1 = 0,$$

$$4x - y + 7 = 0.$$

Analytic Solution: It is simpler to eliminate y than x in this problem, for y occurs only to the first power. Hence, we may solve the linear equation for y and substitute into the quadratic, or, more efficiently in this instance, we add the two equations member by member. This addition gives

$$2x^2 + 8x + 8 = 0,$$

an equation which has real, equal roots. By solving this equation we find $x = -2$. The value $x = -2$ substituted into the linear equation gives $y = -1$. The required solution is $(-2, -1)$. *Ans.*

Graphical Solution: The given equations represent a parabola and a straight line, respectively. The graphs are shown in Figure 86. The straight line is tangent to the curve at the point $(-2, -1)$. *Ans.*

The fact that the straight line is tangent to the conic in Example 2 is characterized by the occurrence of a quadratic equation in one unknown with real, equal roots. (Compare the discussion on pages 150 to 151.)

EXERCISES 103

Solve each of the systems in 1 to 12 both graphically and analytically. Label points of intersection as in the two preceding figures.

1. $9x - y = 6$
$3x^2 = y$

2. $3x - 2y = 9$
$y^2 = 9x$

3. $x - 2y = 10$
$y = x^2 + 2x - 15$

4. $x - y + 2 = 0$
$x = y^2 - 7y + 10$

5. $u^2 + v^2 = 20$
$u + v = 6$

6. $t^2 + u^2 = 29$
$t - 2u = 1$

7. $2s - t = 1$
$st = 15$

8. $3x + 2y + 12 = 0$
$xy + 18 = 0$

9. $5u - 3v = 10$
$u^2 - v^2 = 6$

10. $s = r^2 - 5r + 4$
$5r - s = 21$

11. $x^2 + 3y^2 = 12$
$x - y + 4 = 0$

12. $2x^2 + 3y^2 = 35$
$2x + y = 7$

Solve each of the following systems analytically:

13. $3xy - x^2 = 26$
$2x - y + 1 = 0$

14. $3y^2 + xy = 4$
$x - 3y + 10 = 0$

15. $r^2 + s^2 + 6r - 4s = 12$
$r - 7s = 8$

16. $x^2 + y^2 - 2x - 8y = 83$
$x + y = 19$

17. $3x^2 + 4y^2 = 4$
$2x + 2y = 3$

18. $4u^2 - 3v^2 = -11$
$2u - v = 1$

19. $2kx - 3my = 3k - 3m$
$2xy - 3 = 0$

20. $16c^2e^2x^2 - c^2e^2y^2 = 4c^4 - 4e^4$
$4cex + cey = 2c^2 - 2e^2$

127. Two Quadratics with Square Terms Only

A system of quadratic equations, such as the following:

$$a_1x^2 + b_1y^2 = c_1,$$

$$a_2x^2 + b_2y^2 = c_2,$$

may be solved by any of the methods available for the solution of two linear equations in two unknowns. For, if we let $x^2 = X$ and $y^2 = Y$, we have

$$a_1X + b_1Y = c_1,$$

and

$$a_2X + b_2Y = c_2,$$

which is a system of linear equations in X and Y. When X and Y are found, the desired values of x and y may be obtained by taking square roots, paying due attention to the pairing of the signed values as shown in the next example.

Example 1. Solve analytically the system

$$9x^2 + 4y^2 = 261,$$

$$x^2 - y^2 = 16.$$

Solution: Multiply both members of the second equation by 4, and add the result, member by member, to the first equation to obtain

$$13x^2 = 325.$$

Hence,

$$x^2 = 25$$

and

$$x = \pm 5.$$

If either $+5$ or -5 is substituted for x in the second of the given

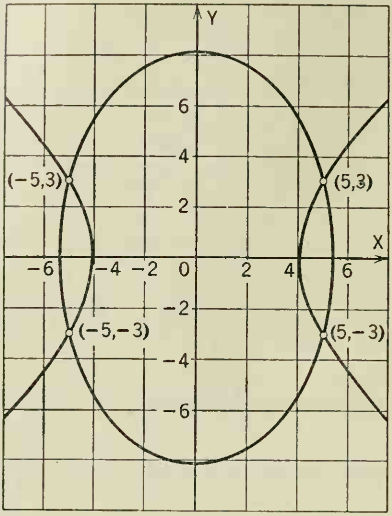


Fig. 87

equations, we find

$$y^2 = 9$$

and $$y = \pm 3.$$

Since only square terms occur in the system of equations, all possible combinations of the plus and minus signs may be used. Therefore, the required solutions are (5, 3), (5, −3), (−5, 3), and (−5, −3). *Ans.*

The graphs of the two equations in Example 1 are shown in Figure 87. This figure is an illustration of the case where two conic sections have four points of intersection, the maximum number possible. It is of interest to note that two curves whose equations are of degrees m and n, respectively, may intersect in mn points at most.

Example 2. Solve analytically the system

$$9x^2 + 25y^2 = 225,$$

$$x^2 + y^2 = 4.$$

Solution: Multiply both sides of the second equation by 9, and subtract the result, member by member, from the first equation to find

$$16y^2 = 189$$

and $$y = \pm \frac{3\sqrt{21}}{4}.$$

If these values of y are substituted into the second equation of the system, there results

$$x^2 = -\frac{125}{16}$$

and $$x = \pm \frac{i5\sqrt{5}}{4}.$$

Therefore, the required solutions are

$$\left(i\frac{5\sqrt{5}}{4}, \frac{3\sqrt{21}}{4}\right), \left(i\frac{5\sqrt{5}}{4}, -\frac{3\sqrt{21}}{4}\right), \left(-i\frac{5\sqrt{5}}{4}, \frac{3\sqrt{21}}{4}\right),$$

and $$\left(-i\frac{5\sqrt{5}}{4}, -\frac{3\sqrt{21}}{4}\right). \quad \textit{Ans.}$$

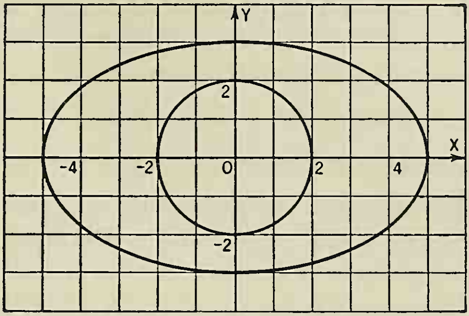


Fig. 88

The graphs of the two equations in Example 2 appear in Figure 88. Since no provision has been made for graphing points with imaginary coordinates, the solutions obtained are not represented in the graph, that is, the curves do not intersect.

EXERCISES 104

Solve each of the systems in 1 to 6 both graphically and analytically. Label points of intersection on your graphs.

1. $x^2 + y^2 = 34$
$x^2 - y^2 = 16$

2. $x^2 + y^2 = 52$
$y^2 - x^2 = 20$

3. $4x^2 + y^2 = 25$
$x^2 + 4y^2 = 40$

4. $2r^2 + s^2 = 72$
$r^2 + s^2 = 36$

5. $5u^2 - 2v^2 = 13$
$3u^2 - v^2 = 11$

6. $4x^2 - 3y^2 = -104$
$4y^2 - 3x^2 = 253$

7. $x^2 + y^2 = 16$
$x^2 - y^2 = -34$

8. $x^2 - 4y^2 = -20$
$3y^2 - x^2 = 19$

Solve each of the following systems analytically:

9. $p^2 - q^2 = 12\frac{1}{2}$
$3p^2 + q^2 = 12\frac{1}{2}$

10. $3x^2 - 2y^2 = 110$
$x^2 - y^2 = 53$

11. $5u^2 + 8v^2 - 106 = 0$
$4u^2 - 5v^2 - 62 = 0$

12. $3r^2 - 5s^2 = 115$
$2r^2 - 7s^2 = 150$

13. $a^2x^2 - b^2y^2 - a^2b^2 = 0$
$b^2x^2 - a^2y^2 + a^2b^2 = 0$

14. $b^2x^2 - a^2y^2 - a^2b^2 = 0$
$x^2 + y^2 - r^2 = 0$

128. The General Method for Systems Involving Quadratics

In solving any system of equations, we rely on the fundamental idea that the solutions consist of sets of numbers which satisfy all of the equations simultaneously. Thus, for the system of quadratics

$$A_1x^2 + B_1xy + C_1y^2 + D_1x + E_1y + F_1 = 0, \tag{1}$$

$$A_2x^2 + B_2xy + C_2y^2 + D_2x + E_2y + F_2 = 0, \tag{2}$$

the solutions would consist of the pairs of coordinates of the points of intersection of the two curves represented by the equations. Since at each point of intersection, x is the same number in both equations and so also is y, we may combine the equations to eliminate either x or y, thereby obtaining one equation in one unknown. The roots of this equation may be substituted into one of the given equations to find the corresponding values of the second unknown. All solutions must, of course, be checked in both of the original equations.

A general method of elimination for Equations (1) and (2) proceeds as follows: If the x^2 and y^2 terms are present in both equations, a third equation can be obtained with the x^2 term missing by multiplying Equation (1) by A_2 and Equation (2) by A_1 and subtracting, member by member, one of the resulting equations from the other. We thus find an equation of the type

$$B_3xy + C_3y^2 + D_3x + E_3y + F_3 = 0, \tag{3}$$

where the new coefficients are simple combinations of the given coefficients. Equation (3) is linear in x and yields

$$x = -\frac{C_3y^2 + E_3y + F_3}{B_3y + D_3}. \tag{4}$$

In case the x^2 (or y^2) term is missing from one of the given equations, the first elimination may be omitted, as an equation of the type of (3) is already at hand. If we substitute this expression for x into either of Equations (1) or (2) and multiply both members by the LCD of the resulting fractions, the final equation will, in general, be of the fourth degree in y, but may, under special circumstances, be of lower degree.

Once the roots of this equation in y are found, they may be substituted into Equation (4) to find the corresponding values of x.

The general solution of a system of two quadratic equations will not be considered in this book. However, if the result of the elimination previously described is a quadratic equation or even an equation of quadratic type, the methods of quadratic equations are available. In any case, if the coefficients of the given equations are numerical, the methods of the preceding chapter are applicable.

Example 1. Find the solutions of the system of equations

$$x^2 - y = 0,$$
$$2x^2 + 3x - y - 4 = 0.$$

Solution: Since the equations are linear in y, we may solve either of them for y and substitute into the other. We choose the first equation, since it is the simpler, and solve it for y to find

$$y = x^2.$$

This result is substituted into the second equation to obtain

$$2x^2 + 3x - x^2 - 4 = 0,$$

or

$$x^2 + 3x - 4 = 0,$$

an equation whose roots are $x = -4$ and $x = 1$. Substituting $x = -4$ into the equation $y = x^2$, we have $y = 16$; similarly, if $x = 1$, $y = 1$. Thus, the solutions of the system are $(-4, 16)$ and $(1, 1)$. *Ans.*

Example 2. Obtain the solutions of the system

$$x^2 + 3y^2 = 7,$$
$$xy = 2.$$

Solution: From the second equation, we have

$$x = \frac{2}{y},$$

which we substitute into the first equation to get

$$\frac{4}{y^2} + 3y^2 = 7.$$

Both members of this equation may be multiplied by y^2 and the terms collected with the result,

$$3y^4 - 7y^2 + 4 = 0,$$

an equation which is quadratic in y^2. We now find $y^2 = 1$ or $\frac{4}{3}$ and $y = \pm 1$ or $y = \pm\frac{2\sqrt{3}}{3}\cdot$ By employing the equation $x = \frac{2}{y}$, we obtain the solutions $(2, 1)$, $(-2, -1)$, $\left(\sqrt{3}, \frac{2\sqrt{3}}{3}\right)$, and $\left(-\sqrt{3}, -\frac{2\sqrt{3}}{3}\right)\cdot$ *Ans.*

EXAMPLE 3. Solve the system of equations

$$x^2 + y^2 = 5,$$
$$xy + 4y = 6.$$

Solution: We solve the second equation for x to obtain

$$x = \frac{6 - 4y}{y}\cdot$$

The result of substituting this expression for x into the first equation and simplifying is

$$y^4 + 11y^2 - 48y + 36 = 0.$$

This equation is found to have the rational roots $y = 1$ and $y = 2$; and the depressed equation is

$$y^2 + 3y + 18 = 0,$$

which has the roots

$$\frac{-3 \pm i3\sqrt{7}}{2}\cdot$$

By using the relation $x = \frac{6 - 4y}{y}$, we find the solutions

$$(2, 1),\ (-1, 2),\ \left(\frac{-9 - i\sqrt{7}}{2}, \frac{-3 + i3\sqrt{7}}{2}\right),$$

and

$$\left(\frac{-9 + i\sqrt{7}}{2}, \frac{-3 - i3\sqrt{7}}{2}\right)\cdot \quad \textit{Ans.}$$

These solutions may be checked in the given equations.

The student should notice in the preceding examples that after the values of one unknown have been found, the corresponding values of the second unknown are obtained by substitution into the most convenient equation which occurs in the procedure. It is frequently simpler to use an intermediate equation rather than one of the original equations.

Although the general method of elimination given in this section may always be used for a system of quadratics, the student should be aware of the possibility of simplifying the procedure in special cases. For example, the system

$$3x^2 - xy = 18.$$

$$x^2 - y^2 = 0,$$

is readily solved by factoring the second equation. The solutions may then be found by solving the two systems

$$\begin{aligned} 3x^2 - xy &= 18, \\ x + y &= 0; \end{aligned} \quad \text{and} \quad \begin{aligned} 3x^2 - xy &= 18, \\ x - y &= 0. \end{aligned}$$

EXERCISES 105

Each of the following systems is to be solved both graphically and analytically. Problems 1 to 6 lead to quadratic or quadratic-type equations in one unknown; Problems 7 to 10 lead to cubics or quartics with rational roots.

1. $x^2 + y^2 = 50$
$xy - 7 = 0$

2. $xy + 6 = 0$
$x^2 + 2y^2 = 17$

3. $u^2 + 5v^2 = 16$
$u^2 + v^2 = 4u$

4. $r^2 - s^2 = 20$
$9s = r^2$

5. $x^2 + y^2 = 40$
$4x = y^2 - 28$

6. $3x^2 - y^2 = 11$
$5y = 3x^2 - 7$

7. $y = x^2 - x - 14$
$xy + 24 = 0$

8. $4v = u^2 + 10u + 8$
$uv - 16 = 0$

9. $6t = 2s^2 + 5s - 24$
$5s = 24 + 6t - 8t^2$

10. $6x = 6 - 7y - y^2$
$y = x^2 - 2x - 6$

Solve each of the following systems analytically:

11. $(2x - y - 1)(2x + y) = 0$
$3x^2 - 3xy + y^2 = 13$

12. $(v + 2w + 5)(5v - 7w) = 0$
$v^2 - 2vw + 3w^2 = 6$

13. $t^2 - 4tu + 3u^2 = 14$
$4u - 2t = 9$

14. $x^2 + xy - 2y^2 = 7$
$x + 3y = -1$

15. $4x^2 + 4xy + y^2 = 81$
$4x^2 - 4xy + y^2 = 1$

16. $r^2 + 5rs - s^2 = 145$
$2s^2 - 3rs = -45$

17. $(5n + p)^2 - 16 = 0$
$3n^2 + 4np + p^2 = 0$

18. $x^2 - 6xy + 9y^2 - 16 = 0$
$6xy + 2y^2 - 9 = 0$

19. $x^2 + y^2 = 10$
$xy - 3x = -6$

20. $r^2 + 2s^2 = 6$
$rs - s = 1$

21. $v^2 - 2vw = -9$
$4vw + w^2 = 45$

22. $2x^2 + 3xy = 5$
$xy + 2y^2 = 3$

23. $x^2 + 4xy + 5y^2 = 20$
$3x^2 + 8xy + 7y^2 = 28$

24. $2t^2 + 8tu + 10u^2 = 53$
$4t^2 + 4tu - 4u^2 = -29$

(HINT: For Problems 25 to 28, divide the members of the first equation of the system by the corresponding members of the second equation.)

25. $r^3 - s^3 = 296$
$r - s = 2$

26. $x^3 + y^3 = 341$
$x + y = 11$

27. $x^2 + xy + y^2 = 133$
$x - \sqrt{xy} + y = 7$

28. $16v^4 + 16v^2w^2 + 16w^4 = 91$
$4v^2 + 4vw + 4w^2 = 13$

The remaining problems of this set may be simplified by first making an appropriate substitution in each case. For example, in Problem 29, we may let $1/x = u$ and $1/y = v$.

29. $\dfrac{1}{x^2} + \dfrac{4}{xy} - \dfrac{3}{y^2} = 9$
$\dfrac{1}{x} + \dfrac{1}{y} = 3$

30. $\dfrac{2}{r^2} - \dfrac{2}{rs} - \dfrac{1}{s^2} = 71$
$\dfrac{1}{r} + \dfrac{1}{s} = -2$

31. $t^{1/4} + u^{1/5} = 5$
$t^{1/2} + u^{2/5} = 13$

32. $x^{1/3} + y^{1/6} = 9$
$x^{2/3} + y^{1/3} = 41$

129. Miscellaneous Methods

The methods considered in the preceding sections are sufficient to enable the student to handle many of the systems of quadratic equations that he is likely to meet in his later work. However, there are also a number of cases in which the solution may be simplified with the aid of special devices. Examples of such devices are given below.

(1) *Homogeneous Quadratic Equations.* Equations whose variable terms are all of the second degree in the variables are called **homogeneous** quadratics. Systems of two homogeneous quadratics can *always* be solved by the method illustrated in the following example.

EXAMPLE 1. Solve the system of equations

$$x^2 - 2xy + 2y^2 = 5, \tag{1}$$

$$x^2 + xy = 15. \tag{2}$$

Solution: Since the terms in x and y are all quadratic, the equations may be combined to obtain a third equation in which the constant term is missing and which is of the form $ax^2 + bxy + cy^2 = 0$. This equation may be solved for x in terms of y (or y in terms of x) and, thus, will usually lead to a pair of linear equations, each of which may be solved simultaneously with either of the given equations.

Following this idea, we multiply both members of Equation (1) by 3 and subtract Equation (2), member by member, from the result. This procedure leads to the equation

$$2x^2 - 7xy + 6y^2 = 0,$$

or

$$(2x - 3y)(x - 2y) = 0. \qquad (3)$$

Since Equation (3) is equivalent to the two equations

$$2x - 3y = 0 \quad \text{and} \quad x - 2y = 0,$$

we may combine them with Equation (2) to yield the two systems:

$$\left.\begin{aligned} x^2 + xy &= 15, \\ 2x - 3y &= 0; \end{aligned}\right\} \qquad (4)$$

and

$$\left.\begin{aligned} x^2 + xy &= 15, \\ x - 2y &= 0. \end{aligned}\right\} \qquad (5)$$

Systems (4) and (5) each consists of one linear and one quadratic equation and, accordingly, may be solved by the method of Section 126. Hence, the solutions of the given system consist of

the solutions of (4), namely, $(3, 2)$ and $(-3, -2)$,

and the solutions of (5), namely,

$$(\sqrt{10}, \tfrac{1}{2}\sqrt{10}) \quad \text{and} \quad (-\sqrt{10}, -\tfrac{1}{2}\sqrt{10}).$$

(2) *Symmetric Equations.* An equation that remains unaltered if x and y are interchanged is said to be **symmetric in x and y.** Thus,

$$x^2 + 3xy + y^2 - x - y = 5$$

is symmetric in x and y.

A system of two symmetric quadratics in x and y may be solved by substituting two new unknowns u and v, which are related to x and y by means of the equations

$$x = u + v,$$

and

$$y = u - v.$$

This method is illustrated in the next example.

Example 2. Solve the system

$$x^2 + y^2 - x - y = 18, \qquad (1)$$

$$xy + 2x + 2y = 26. \qquad (2)$$

Solution: We set $x = u + v$ and $y = u - v$,

and substitute these expressions into Equations (1) and (2). Then, Equation (1) becomes

$$2u^2 + 2v^2 - 2u = 18,$$

or

$$u^2 + v^2 - u = 9. \qquad (3)$$

Also, Equation (2) becomes

$$u^2 - v^2 + 4u = 26. \qquad (4)$$

By adding the corresponding members of (3) and (4), we find

$$2u^2 + 3u = 35,$$

an equation whose roots are $u = -5$ and $u = +\frac{7}{2}$. By substituting these values into Equation (3), we find if $u = -5$, $v = \pm i\sqrt{21}$, and if $u = +\frac{7}{2}$, $v = \pm\frac{1}{2}$. The solutions of the given system may be displayed as follows:

u	v	$x = u + v$	$y = u - v$
-5	$i\sqrt{21}$	$-5 + i\sqrt{21}$	$-5 - i\sqrt{21}$
-5	$-i\sqrt{21}$	$-5 - i\sqrt{21}$	$-5 + i\sqrt{21}$
$\frac{7}{2}$	$\frac{1}{2}$	4	3
$\frac{7}{2}$	$-\frac{1}{2}$	3	4

Although there are many more special devices of the type discussed in this section, they are of relatively minor importance and will not be considered here. The student should not lose sight of the fact that the general method of elimination is always available. This means that real solutions of systems of quadratics in two unknowns with numerical coefficients may always be found as accurately as may be desired.

EXERCISES 106

Solve each of the following systems by the method for homogeneous equations:

1. $x^2 + 3xy = 10$
$8xy - y^2 = 15$

2. $2x^2 + xy = 12$
$2y^2 + 3xy = -10$

3. $5xy - x^2 = 18$
$2y^2 + 5xy = 36$

4. $2u^2 + 5uv = 3$
$v^2 - 2uv = 7$

5. $17rs - 12r^2 = 56$
$3s^2 - 5rs = 28$

6. $4x^2 - xy = 8$
$y^2 + 2xy = 32$

7. $x^2 - xy = 2$
$4x^2 - 2xy - 3y^2 = 9$

8. $t^2 - tu - u^2 = -5$
$2t^2 + 3tu + u^2 = 2$

Solve each of the following systems by the method for symmetric equations:

9. $x^2 + xy + y^2 = 19$
$xy - x - y = 1$

10. $2x^2 + 2y^2 = 25$
$4xy - 2x - 2y = 15$

11. $x^2 + y^2 + 3xy + x + y = 2$
$x^2 + y^2 - 7x - 7y = 8$

12. $x^2 + y^2 + xy - 12x - 12y = 36$
$xy - 4x - 4y = 0$

EXERCISES 107

1. Find two numbers such that their difference is 3 and the difference of their cubes is 513.

2. Find two numbers such that their product is 176 and the sum of their squares is 377.

3. Find two positive numbers such that their product, their difference, and the difference of their squares are all equal.

4. The diagonal of a rectangle is 34 ft and the perimeter is 92 ft. Find the dimensions of the rectangle.

5. Find the lengths of the sides of a rectangle whose area is 96 sq in. and whose perimeter is 48 in.

6. The perimeter of an isosceles triangle is 50 in. and the area 120 sq in. Find the side opposite the vertex angle.

7. A transport plane flew 600 miles at its normal speed; but on account of motor trouble, it returned to its base at a speed 60 mph slower than normal. If it took 50 min longer on the return than on the outbound flight, what was its normal speed?

8. A person receives \$170 interest from an amount loaned at simple interest. If the interest rate had been 1 per cent higher, he would have received \$238. What was the amount loaned and what was the interest rate?

9. A parking lot for cars has an area of 27,000 sq ft. If a strip 10 ft wide is eliminated on each of the ends and a strip 10 ft wide is eliminated on each of the sides, the available parking space is 20,800 sq ft. What are the over-all dimensions of the original parking lot?

10. When the third term of a geometric progression is subtracted from the first, the result is 630; when the fourth term is subtracted from the second term the result is 105. Find the first four terms of the progression.

11. If each side of the square base of a rectangular box is increased by 3 in. and the altitude is increased by 5 in., the volume is increased by 540 cu in. and the total surface area is increased by 306 sq in. Find the original dimensions of the box.

12. A plane flies a certain distance at a uniform speed. If it had flown 50 mph faster, it would have arrived 1 hr sooner. If it had flown 30 mph

slower, it would have taken 45 min longer. Find the speed of the plane and the distance flown.

13. Two planes fly a distance of 3000 miles, and the faster plane arrives $2\frac{1}{2}$ hr ahead of the slower one. If the speed of each plane had been increased 100 mph, the faster plane would have arrived $1\frac{1}{2}$ hr before the slower one. Find the speed of each plane.

14. The hypotenuse of a right triangle is 5 in. longer than the shorter leg and $6\frac{1}{2}$ in. longer than the altitude drawn to the hypotenuse. Find the length of each leg of the right triangle.

Chapter 19

DETERMINANTS

130. The Solution of Two Linear Equations

In this chapter, we shall need the general solution of a system of two linear equations in two unknowns. To obtain this solution, consider the equations

$$a_1x + b_1y = c_1, \tag{1}$$

$$a_2x + b_2y = c_2. \tag{2}$$

In order to eliminate y, we multiply both sides of Equation (1) by b_2 and both sides of Equation (2) by b_1, and then subtract the second resulting equation, member by member, from the first. This procedure gives

$$a_1b_2x - a_2b_1x = c_1b_2 - c_2b_1,$$

or

$$x(a_1b_2 - a_2b_1) = c_1b_2 - c_2b_1.$$

Hence, if the coefficient of x is not zero,

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}. \tag{3}$$

In a similar way, x may be eliminated from the given system to obtain

$$a_1b_2y - a_2b_1y = a_1c_2 - a_2c_1,$$

or

$$y(a_1b_2 - a_2b_1) = a_1c_2 - a_2c_1.$$

Therefore, if the coefficient of y is not zero,

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}. \tag{4}$$

An inspection of Equations (3) and (4) shows that, first, the expres-

sions for x and y have the same denominator, namely, $a_1b_2 - a_2b_1$; second, the numerator for x may be obtained from the denominator by replacing each a by the c with the same subscript; and third, the numerator for y may be obtained from the denominator by replacing each b by the corresponding c. It is therefore not surprising that a very simple scheme can be devised as an aid toward writing the required solution.

131. Solution of Two Linear Equations by Determinants

An array of numbers, enclosed by two vertical bars, such as

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

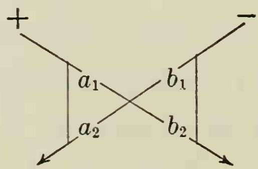
is called a **determinant**; the meaning of this symbol comprising four numbers is defined by the equation

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1.$$

Illustration:

$$\begin{aligned} \begin{vmatrix} 2 & -3 \\ 6 & -7 \end{vmatrix} &= (2)(-7) - (-3)(6) \\ &= -14 + 18 \\ &= 4. \end{aligned}$$

Each number in the array is an **element** of the determinant; thus, a_1, a_2, b_1, and b_2 are the elements of the preceding determinant. Notice that the value of a determinant of four elements consists of the product of the two elements in the diagonal from the upper left to the lower right, minus the product of the two elements in the other diagonal as is indicated in the following diagram:



The preceding determinant is the array of coefficients in their natural positions on the left sides of Equations (1) and (2) in Section 130; the value of the determinant, as just defined, is the denominator in the solution given by the relations (3) and (4) of the same section. We

shall speak of this determinant as *the determinant of the coefficients* and shall denote it by D, that is,

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

Furthermore, the numerator on the right side of relation (3) is the determinant

$$D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix},$$

which may be obtained from D by replacing the coefficients of x, that is, the a's, by the corresponding c's. The subscript x in D_x is to suggest this replacement. Similarly, the numerator on the right side of (4) may be written

$$D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix},$$

which is obtained from D by replacing the coefficients of y, that is, the b's, by the corresponding c's; again, the subscript y in D_y is to suggest this replacement.

We shall take the standard form for a system of two equations in two unknowns as

$$\begin{aligned} a_1x + b_1y &= c_1, \\ a_2x + b_2y &= c_2. \end{aligned} \tag{1}$$

The solution of this system (1) may be obtained as follows: *Evaluate the three determinants* D, D_x, *and* D_y. *Then,*

$$x = \frac{D_x}{D}, \qquad y = \frac{D_y}{D} \qquad (D \neq 0). \tag{2}$$

EXAMPLE 1. Solve by determinants

$$\begin{aligned} 5x + 7y &= 24, \\ 2x - 3y &= 5. \end{aligned}$$

Solution:

$$D = \begin{vmatrix} 5 & 7 \\ 2 & -3 \end{vmatrix} = -15 - 14 = -29;$$

$$D_x = \begin{vmatrix} 24 & 7 \\ 5 & -3 \end{vmatrix} = -72 - 35 = -107;$$

and

$$D_y = \begin{vmatrix} 5 & 24 \\ 2 & 5 \end{vmatrix} = 25 - 48 = -23.$$

Hence, $$x = \frac{D_x}{D} = \frac{-107}{-29} = \frac{107}{29},$$

and $$y = \frac{D_y}{D} = \frac{-23}{-29} = \frac{23}{29}. \quad \textit{Ans.}$$

This answer may be checked in the given equations. *The importance of a check cannot be overemphasized when the solution is obtained in the mechanical way illustrated by this example.*

The student should be careful to keep unchanged the order in which the coefficients in the standard form are used; for it is evident from the definition that

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = -\begin{vmatrix} b_1 & a_1 \\ b_2 & a_2 \end{vmatrix}; \tag{A}$$

thus, *a change in order of the two columns changes the sign of the determinant.* This fact will prove useful to us in a later discussion.

In the general result, preceding Example 1, the provision $D \neq 0$ has been noted. If this condition is violated, we have

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

$$= a_1b_2 - a_2b_1 = 0.$$

This means $$\frac{a_1}{a_2} = \frac{b_1}{b_2}$$

and the two equations of the system are either inconsistent or dependent. (Compare Section 33.) We shall return to this situation later.

EXERCISES 108

Find the value of each of the following determinants:

1. $\begin{vmatrix} 4 & 3 \\ 2 & 7 \end{vmatrix}$ **2.** $\begin{vmatrix} 2 & 7 \\ 5 & 8 \end{vmatrix}$ **3.** $\begin{vmatrix} 3 & 6 \\ -4 & -5 \end{vmatrix}$ **4.** $\begin{vmatrix} 6 & 2 \\ -11 & -5 \end{vmatrix}$

5. $\begin{vmatrix} -5 & -6 \\ 9 & 7 \end{vmatrix}$ **6.** $\begin{vmatrix} 14 & -9 \\ 3 & 0 \end{vmatrix}$ **7.** $\begin{vmatrix} 3a & c \\ -d & 2b \end{vmatrix}$ **8.** $\begin{vmatrix} 9m & -4r \\ -5s & -n \end{vmatrix}$

Use determinants to solve each of the following systems of equations:

9. $5u + 2v = 11$
$3u + 4v = 1$

10. $6r - 7s = 9$
$5r - s = 22$

11. $2x - 3y = 36$
$3x + 2y = 2$

12. $4x + 11y = -6$
$x + 7y = 7$

13. $8x + 2y - 7 = 0$
$10x + 9y - 12 = 0$

14. $3w - z - 4 = 0$
$9w + 7z + 3 = 0$

15. $\frac{r}{5} + \frac{s}{4} - \frac{5}{2} = 0$
$\frac{r}{2} - \frac{s}{5} - \frac{23}{5} = 0$

16. $\frac{3u}{14} + \frac{v}{7} - 4 = 0$
$\frac{2u}{3} + \frac{v}{2} - 7 = 0$

17. $ax - by = c$
$bx + ay = d$

18. $ax + by = 2ab$
$bx + ay = a^2 + b^2,\ a^2 \neq b^2$

19. $3(u + v) - 10(u - v) = 8$
$2(u + v) - 7(u - v) = 5$

20. $2(r + s) + 7(r - s) = 1$
$6(r + s) - 3(r - s) = -5$

21. $\frac{1}{x} - \frac{2}{y} = 4$
$\frac{2}{x} + \frac{8}{y} = -7$

22. $\frac{28}{3w} + \frac{10}{z} = -\frac{5}{3}$
$\frac{4}{w} + \frac{5}{z} = \frac{2}{3}$

23. $\frac{24}{2r + s} + \frac{9}{4r - s} = 5$
$\frac{4}{2r + s} + \frac{5}{4r - s} = 2$

24. $\frac{7}{3x + 4y} + \frac{11}{4x - 3y} = 6$
$\frac{21}{3x + 4y} - \frac{15}{4x - 3y} = -6$

132. Further Properties of a Determinant

We saw in the preceding section that the interchange of the two vertical columns of the determinant

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

causes a change in the sign of the value of the determinant. It should be clear that the same effect is obtained by interchanging the two horizontal rows.

Another simple property of the determinant is given by the identity

$$\begin{vmatrix} ma_1 & b_1 \\ ma_2 & b_2 \end{vmatrix} = m \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \tag{B}$$

which is easily verified by evaluating both members. Thus, *a common factor of each element of one column (or row) may be removed and written as a multiplier of the resulting determinant.* It may be noted in particular that m may have the value -1.

Direct evaluation may be used to verify a third property, namely,

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + \begin{vmatrix} k_1 & b_1 \\ k_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 + k_1 & b_1 \\ a_2 + k_2 & b_2 \end{vmatrix}. \tag{C}$$

The sum of two determinants, each with four elements and having a common column, may be written in the form of a single determinant with the common column retained and with a second column whose elements are the sums of the corresponding elements in the unlike columns.

133. Three Linear Equations in Three Unknowns

For the standard form of a system of three linear equations in three unknowns, we take

$$a_1x + b_1y + c_1z = d_1, \tag{1}$$

$$a_2x + b_2y + c_2z = d_2, \tag{2}$$

$$a_3x + b_3y + c_3z = d_3. \tag{3}$$

As a first step in the solution of this system we may eliminate y and z to obtain a single equation in x. It is instructive to perform this elimination by means of determinants. We rewrite Equations (2) and (3) in the form

$$b_2y + c_2z = d_2 - a_2x,$$

and

$$b_3y + c_3z = d_3 - a_3x,$$

which we may regard as a system of two equations in the two unknowns y and z. We solve this system for y and z in terms of x to obtain

$$y = \frac{\begin{vmatrix} d_2 - a_2x & c_2 \\ d_3 - a_3x & c_3 \end{vmatrix}}{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}},$$

and

$$z = \frac{\begin{vmatrix} b_2 & d_2 - a_2x \\ b_3 & d_3 - a_3x \end{vmatrix}}{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}}.$$

Using the properties discussed in the preceding sections, we have

$$\begin{vmatrix} d_2 - a_2x & c_2 \\ d_3 - a_3x & c_3 \end{vmatrix} = \begin{vmatrix} d_2 & c_2 \\ d_3 & c_3 \end{vmatrix} + \begin{vmatrix} -a_2x & c_2 \\ -a_3x & c_3 \end{vmatrix} \qquad \text{[By (C)]}$$

$$= \begin{vmatrix} d_2 & c_2 \\ d_3 & c_3 \end{vmatrix} - x\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}. \qquad \text{[By (B)]}$$

Similarly,

$$\begin{vmatrix} b_2 & d_2 - a_2x \\ b_3 & d_3 - a_3x \end{vmatrix} = \begin{vmatrix} b_2 & d_2 \\ b_3 & d_3 \end{vmatrix} + \begin{vmatrix} b_2 & -a_2x \\ b_3 & -a_3x \end{vmatrix}$$

$$= -\begin{vmatrix} d_2 & b_2 \\ d_3 & b_3 \end{vmatrix} + x\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}. \quad \text{[By (A) and (B)]}$$

We now substitute the expressions for y and z into Equation (1) making use of the transformed numerators just obtained. The result is

$$a_1x + \frac{b_1\begin{vmatrix} d_2 & c_2 \\ d_3 & c_3 \end{vmatrix} - b_1x\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}}{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}} + \frac{-c_1\begin{vmatrix} d_2 & b_2 \\ d_3 & b_3 \end{vmatrix} + c_1x\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}} = d_1.$$

By multiplying both members of this equation by the determinant in the denominators and isolating on the left side the terms involving x, we find

$$x\left\{a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}\right\}$$

$$= \left\{d_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} d_2 & c_2 \\ d_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} d_2 & b_2 \\ d_3 & b_3 \end{vmatrix}\right\}. \tag{4}$$

Notice carefully that the second expression in braces may be obtained from the first by replacing the a's throughout by the corresponding d's. The coefficient of x may be obtained from the array

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

by following the procedure explained in the next paragraph.

The coefficient of the element a_1 is the determinant obtained by crossing out the row and column to which a_1 belongs; the coefficient of b_1 is, except for sign, the determinant obtained by crossing out the row and column to which b_1 belongs; and similarly for the coefficient of c_1. Each of these coefficients is a determinant of two rows and two columns and is spoken of as a **second-order determinant.** The array of three rows and three columns is called a **third-order determinant.**

Thus, let us define

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}.$$

The second-order determinant by which a_1 is multiplied is called the **minor of a_1** and similarly for the remaining two second-order determinants. There are many ways of defining a third-order determinant that give a result equivalent to the one just described; this particular method of evaluating a third-order determinant is an **expansion by minors.**

As in the case of two equations in two unknowns, we write D for the determinant of the coefficients of a system of three linear equations in three unknowns. Also D_x will stand for the determinant obtained by replacing the coefficients of x by the corresponding constant terms. A similar convention will be used with respect to D_y and D_z. With this understanding, we return to Equation (4) which may be written in the form

$$xD = D_x.$$

Hence, if $D \neq 0$,
$$x = \frac{D_x}{D}.$$

We may obtain the values of y and z in the same way as that used to find the value of x, or we may regard the results as evident from the symmetrical way in which x, y, z, and their respective coefficients enter the equations. Thus, we have

$$y = \frac{D_y}{D} \quad \text{and} \quad z = \frac{D_z}{D}.$$

Example 1. Evaluate the determinant

$$\begin{vmatrix} 2 & -1 & -3 \\ 5 & 9 & 4 \\ 7 & 6 & -2 \end{vmatrix}.$$

Solution: We expand by minors to find as the desired value

$$2\begin{vmatrix} 9 & 4 \\ 6 & -2 \end{vmatrix} - (-1)\begin{vmatrix} 5 & 4 \\ 7 & -2 \end{vmatrix} + (-3)\begin{vmatrix} 5 & 9 \\ 7 & 6 \end{vmatrix}$$
$$= 2(-18 - 24) + (-10 - 28) - 3(30 - 63) = -23. \quad \textit{Ans.}$$

Example 2. Solve the following system by means of determinants:

$$2x + 3y + z = 4,$$
$$x + 5y - 2z = -1,$$
$$3x - 4y + 4z = -1.$$

Solution:

$$D = \begin{vmatrix} 2 & 3 & 1 \\ 1 & 5 & -2 \\ 3 & -4 & 4 \end{vmatrix} = 2(12) - 3(10) + 1(-19) = -25;$$

$$D_x = \begin{vmatrix} 4 & 3 & 1 \\ -1 & 5 & -2 \\ -1 & -4 & 4 \end{vmatrix} = 4(12) - 3(-6) + 1(9) = 75;$$

$$D_y = \begin{vmatrix} 2 & 4 & 1 \\ 1 & -1 & -2 \\ 3 & -1 & 4 \end{vmatrix} = 2(-6) - 4(10) + 1(2) = -50;$$

and $$D_z = \begin{vmatrix} 2 & 3 & 4 \\ 1 & 5 & -1 \\ 3 & -4 & -1 \end{vmatrix} = 2(-9) - 3(2) + 4(-19) = -100.$$

Therefore, $$x = \frac{D_x}{D} = -3,$$

$$y = \frac{D_y}{D} = +2,$$

and $$z = \frac{D_z}{D} = +4. \quad \textit{Ans.}$$

EXERCISES 109

Evaluate each of the following determinants:

1. $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 3 & 5 \\ 7 & 1 & 2 \end{vmatrix}$ **2.** $\begin{vmatrix} 8 & 5 & 9 \\ 7 & 6 & 1 \\ 3 & 4 & 2 \end{vmatrix}$ **3.** $\begin{vmatrix} -3 & 1 & 2 \\ 1 & -1 & -4 \\ 2 & 3 & 5 \end{vmatrix}$

4. $\begin{vmatrix} 6 & 7 & 1 \\ 0 & 3 & 3 \\ 4 & 1 & -5 \end{vmatrix}$ **5.** $\begin{vmatrix} 8 & 6 & 7 \\ 3 & 4 & 0 \\ -2 & -1 & 3 \end{vmatrix}$ **6.** $\begin{vmatrix} d & 0 & d \\ c & d & c \\ 0 & c & d \end{vmatrix}$

7. $\begin{vmatrix} 1 & -1 & b \\ -1 & b & -1 \\ b & -1 & -1 \end{vmatrix}$ **8.** $\begin{vmatrix} u & v & 1 \\ 3 & 2 & 1 \\ 7 & 6 & 1 \end{vmatrix}$ **9.** $\begin{vmatrix} x & y & 1 \\ 5 & -1 & 1 \\ 2 & -3 & 1 \end{vmatrix}$

Solve each of the following systems of equations by determinants:

10. $x + 2y - z = 1$
$4x - y + 2z = -17$
$3x + y - 4z = -13$

11. $2r + 3s - t = -1$
$3r + 4s + 2t = 14$
$r - 6s - 5t = 4$

12. $4u - v - 3w = 5$
$6u + 2v + 7w = 1$
$3u - 3v - 8w = 3$

13. $5x + 2y + 4z = -5$
$7x + 8y - 2z = 13$
$2x - 5y + 3z = 4$

14. $2x - 5y + 2z = 8$
$4x + 6y - z = -2$
$2x + 7y + 4z = 6$

15. $3u + 4v + 6w = 11$
$u + 8v - 2w = 12$
$3u + 6v - 9w = 2$

16. $5r + 3s = -1$
$9s + 8t = 13$
$16t - 15r = 23$

17. $7x + 18y = 7$
$5y - 2z = 10$
$3z + 10x = -5$

18. $gx - ey = f$
$fy + gz = e$
$ez - fx = g$

19. $ax + 2by = c$
$2cy + az = b$
$bz + cx = a$

20. $\frac{2}{u} - \frac{3}{v} + \frac{1}{w} = 22$
$\frac{5}{u} + \frac{6}{v} + \frac{4}{w} = 22$
$\frac{1}{u} - \frac{2}{v} = 10$

21. $\frac{3}{r} + \frac{8}{s} - \frac{4}{t} = 3$
$\frac{6}{r} + \frac{2}{s} + \frac{3}{t} = -3$
$\frac{9}{r} - \frac{6}{s} - \frac{5}{t} = -2$

134. Higher-Order Determinants

We shall first rewrite the expansion of a third-order determinant using a notation which is better suited for the consideration of determinants of higher order. In this notation, each element is supplied with a double subscript, the first subscript indicating the number of the row, and the second the number of the column to which the element belongs. Thus, a_{32} (read "a sub three-two" or "a-three-two") is the symbol for the element in the third row and the second column. Using double subscripts, we have

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}.$$

An inspection of this result shows that it contains all the products that can be formed from the given array by taking one, and only one, element from each row and column.

In order to provide a law that will explain the sign preceding each product, we note first that the row subscripts have all been written in natural numerical order: 1, 2, 3; whereas the column subscripts are not in natural order except for the first product. In a sequence of subscripts, if an integer precedes a smaller integer, we say there is an **inversion**. Thus, 2, 3, 1 contains two inversions; for 2 precedes 1 and 3 precedes 1.

Let us list the number of inversions in the column subscripts for the preceding products as follows:

	Column Subscripts	Number of Inversions
Plus Products	1, 2, 3	0
	2, 3, 1	2
	3, 1, 2	2
Minus Products	1, 3, 2	1
	2, 1, 3	1
	3, 2, 1	3

We see now that, *if there is no inversion or if there is an even number of inversions, the sign prefixed to the product is plus; if there is an odd number of inversions, the sign is minus; consequently, the correct sign will always be given by* $(-1)^{\mathrm{k}}$, *where* k *stands for the number of inversions.*

In accordance with this discussion, we call the following array of n rows and n columns a **determinant of the *n*th order**:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}$$

The value of this determinant is defined as the algebraic sum of all the signed products which can be formed by taking one, and only one, element from each row and column. The sign to be prefixed to each product is $(-1)^{\mathrm{k}}$, *where* k *is the number of inversions in the column subscripts when the factors of the product are arranged with their row subscripts in natural order.*

In writing one of the products in the evaluation of an nth-order determinant, there are n ways of choosing the first factor; after this choice is made, there are $n-1$ ways of choosing the second factor; then $n-2$ ways of choosing the third factor, and so on. Since each combination so chosen gives a term of the final sum, we see that there are

$$n(n-1)(n-2)\cdots 2\cdot 1 = n!$$

terms in the expansion. For example, we have seen that a third-order determinant has $3! = 6$ terms; a fourth-order determinant would have $4! = 24$ terms; and a fifth-order determinant would have $5! = 120$ terms.

Because of the large number of terms involved, it is not practical to evaluate a higher-order determinant directly by means of the definition. However, easier methods become available once we discover a few fundamental properties of determinants.

We note that in obtaining the proper sign for one of the products in the preceding definition, the factors may be so arranged that the column rather than the row subscripts are in natural order. We then count the number of inversions in the row subscripts to determine the sign as before. It is not difficult to see that this number is the same as the number of inversions originally in the column subscripts. For, the number of inversions in the column subscripts can be found by counting the number of subscripts greater than 1 that precede 1, plus the number of subscripts greater than 2 that precede 2, plus and so on. Suppose that column index 1 occurs in the $(k + 1)$th position rather than in the first, so that its placement accounts for k of the total number of inversions. If the 1 is moved to first place, it carries with it the row index $k + 1$ which then precedes k smaller row indices. Thus there are then k inversions in the row indices and k of the inversions in the column subscripts are removed. Next, if column index 2 is not now in second place but is in place number $m + 2$, its position accounts for m more inversions. If the 2 is moved back to its natural position, then row index $m + 2$ is also put into second place, and will precede m smaller indices. Continuing in this manner, we see that our conclusion with regard to the final number of inversions in the row subscripts is correct.

It follows now that the value of a determinant is unchanged by writing the rows in order as columns. For example,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

From this discussion, there may be drawn an important conclusion, namely,

Property 1: *In any theorem involving a determinant, the words "row" and "column" may be interchanged throughout.*

The following theorem is an extension of a statement made for second-order determinants:

Property 2: *If any two columns (or rows) of a determinant are interchanged, the sign of the determinant is changed.*

The validity of the theorem for the case of two adjacent columns is

seen at once from the fact that two column subscripts are interchanged and the row subscripts are left unaltered. This means that the number of inversions in each product is increased or decreased by 1. Hence, the sign of the determinant is changed.

Now, suppose we wish to interchange the ith and the kth columns, and suppose these two have m other columns between them. The desired interchange may be effected as follows: Make $m + 1$ successive interchanges of two adjacent columns to bring the kth column into the position previously occupied by the ith. The original ith column becomes the $(i + 1)$th column and, with m similar interchanges, may be put into the position previously occupied by the kth column. Since $m + 1 + m = 2m + 1$ is an odd integer, an odd number of interchanges of two adjacent columns is used, and the sign of the determinant is changed.

Property (2) enables us to derive very simply the following theorem, namely:

Property 3: *A determinant with two identical columns (or rows) has the value zero.*

If the value of the determinant is x, the value $-x$ would be obtained by interchanging the two identical columns. But, if these columns are identical, the determinant is unaltered by this interchange. Hence, $x = -x$; from which we obtain $2x = 0$ or $x = 0$.

The next theorem is again an extension of a result stated for second-order determinants.

Property 4: *If each element of any column (or row) is multiplied by the same number* m, *the value of the determinant is multiplied by* m. For example,

$$\begin{vmatrix} ma_1 & b_1 & c_1 \\ ma_2 & b_2 & c_2 \\ ma_3 & b_3 & c_3 \end{vmatrix} = m \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

The proof of Property (4) is obtained directly from the general definition of the value of a determinant. If each element in any column is multiplied by m, each term in the sum that defines this value is multiplied by m. Hence, the value of the determinant is multiplied by m.

By taking $m = 0$ in (4), we find that *if every element of any row or column is zero, the value of the determinant is zero.*

Property 5: *The sum of two determinants, in which each column except one of the first is identical with the corresponding column of the second, may*

be written as a single determinant with the common columns retained; the elements of the remaining column are the sums of the corresponding elements in the two nonidentical columns. For example,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + k_1 & b_1 & c_1 \\ a_2 + k_2 & b_2 & c_2 \\ a_3 + k_3 & b_3 & c_3 \end{vmatrix}.$$

From the definition of the value of a determinant, we see that each term of the expansion of the single determinant is the sum of the two corresponding terms of the other determinants.

We may now obtain a result which will be found useful in the numerical evaluation of determinants.

Property 6: *The value of a determinant is unaltered if to each element of one column (row) there is added* m *times the corresponding element of another column (row).* (Note that m may have any value; in particular, it may be $+1$ or -1.) Thus, using the notation of the earlier portions of this chapter, we should have, for example,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + mb_1 & b_1 & c_1 \\ a_2 + mb_2 & b_2 & c_2 \\ a_3 + mb_3 & b_3 & c_3 \end{vmatrix}.$$

After denoting the determinant on the left side by D and the one on the right side by D_1, we may first use Property (5) to obtain

$$D_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} mb_1 & b_1 & c_1 \\ mb_2 & b_2 & c_2 \\ mb_3 & b_3 & c_3 \end{vmatrix}.$$

Then, if m is factored out of the second determinant, the resulting determinant has two identical columns and is, therefore, zero. Hence, $D_1 = D$. This demonstration for third-order determinants may evidently be used with almost no modification for determinants of higher order. Furthermore, a similar procedure will serve to demonstrate Property (6) for any two columns or any two rows.

EXERCISES 110

1. How many terms are there in the expansion of a determinant of the seventh order? of the ninth order?

2. Give the number of inversions in each of the following sequences:

(*a*) 3, 1, 5, 4, 2	(*b*) 2, 4, 3, 1, 5	(*c*) 5, 3, 4, 2, 1
(*d*) 5, 2, 1, 4, 3	(*e*) 4, 3, 1, 5, 2	(*f*) 1, 5, 4, 2, 3
(*g*) 4, 5, 3, 2, 1	(*h*) 2, 1, 3, 4, 5	(*i*) 3, 5, 1, 4, 2

3. Each of the following products is, except for sign, a term of the expansion of a fourth-order determinant. Find the correct sign in each case.

(*a*) $a_{14}a_{21}a_{33}a_{42}$ (*b*) $a_{13}a_{22}a_{31}a_{44}$ (*c*) $a_{12}a_{24}a_{31}a_{43}$
(*d*) $a_{11}a_{23}a_{32}a_{44}$ (*e*) $a_{13}a_{24}a_{31}a_{42}$ (*f*) $a_{12}a_{23}a_{34}a_{41}$

Show, without direct evaluation, that each of the next two determinants has the value zero. Then, verify by actual evaluation.

4. $\begin{vmatrix} 2 & -3 & 2 \\ 1 & 4 & 1 \\ 9 & 6 & 9 \end{vmatrix}$ **5.** $\begin{vmatrix} 5 & -4 & 6 \\ 7 & 2 & 8 \\ 15 & -12 & 18 \end{vmatrix}$

Before evaluating the following determinants, show that 6 is a factor of each; then expand to verify this fact.

6. $\begin{vmatrix} 6 & 1 & 5 \\ -12 & 2 & 3 \\ 12 & -1 & 1 \end{vmatrix}$ **7.** $\begin{vmatrix} 2 & 8 & 6 \\ 1 & -1 & 3 \\ 4 & 2 & 3 \end{vmatrix}$

Show by expanding that, if the first and third rows are interchanged, the value of each of the following determinants is changed only in sign.

8. $\begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & 2 \\ 3 & 1 & 2 \end{vmatrix}$ **9.** $\begin{vmatrix} 5 & 2 & 1 \\ 1 & 4 & -2 \\ -3 & -1 & 5 \end{vmatrix}$

In each of the next two exercises, use Property (5) to express the sum of the determinants as a single determinant.

10. $\begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & 2 \\ 4 & -3 & 5 \end{vmatrix} + \begin{vmatrix} 3 & 2 & 3 \\ -5 & -1 & 2 \\ -1 & -3 & 5 \end{vmatrix}$ **11.** $\begin{vmatrix} 5 & 1 & -8 \\ 2 & 4 & -3 \\ 1 & -2 & 4 \end{vmatrix} + \begin{vmatrix} -3 & 2 & 5 \\ 2 & 4 & -3 \\ 1 & -2 & 4 \end{vmatrix}$

12. For the adjoining determinant add the third column twice to the second column. Then, show, by direct evaluation, that the value of the determinant is unchanged.

$$\begin{vmatrix} 4 & -2 & 1 \\ 5 & 2 & -3 \\ 2 & -3 & 3 \end{vmatrix}$$

13. For the determinant given in the preceding exercise, subtract the third column four times from the first column. Then, show, by direct evaluation, that the value of the determinant is unchanged.

135. Expansion of a Determinant by Minors

The *minor* of an element of a determinant is the determinant that remains when the column and row containing the given element are deleted. For example, in the following determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix},$$

the minor of a_{12} is $\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$.

The minor of the element a_{ij} (which lies in the ith row and the jth column) will be denoted by M_{ij}. The second-order determinant displayed in the preceding illustration is M_{12}.

We may now show how a determinant of any order may be expanded by using the minors of one column or row.

Let

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix}.$$

We show first that the product of a_{11} and its minor, that is, $a_{11}M_{11}$, contains all the terms which have a_{11} as a factor in the expansion. Clearly, $a_{11}M_{11}$ is the sum of the signed products obtained by multiplying a_{11} by one, and only one, element from each remaining row and column. Furthermore, a_{11} would be the first element written in determining the proper sign for the product. Therefore, the sign of each term in $a_{11} M_{11}$ is the same as that in M_{11} alone. But, the elements of M_{11} are arranged in the same order as in the original determinant; hence, $a_{11}M_{11}$ contains exactly those terms of the expansion which have a_{11} as a factor.

Returning to the determinant D, we may move the ith row to the top by interchanging it successively with the $i - 1$ rows above it. This procedure will produce $i - 1$ changes in sign. Next, we may move the jth column to the first position in a similar fashion, this time producing $j - 1$ changes in sign. If D' is the determinant that results, the element a_{ij} is in the position previously occupied by a_{11}, and

$$D' = (-1)^{i-1+j-1}D = (-1)^{i+j}(-1)^{-2}D,$$

that is, $$D' = (-1)^{i+j}D.$$

Therefore, the terms of the expansion of D that have a_{ij} as a factor are given by

$$(-1)^{i+j}a_{ij}M_{ij},$$

where M_{ij} is the minor of a_{ij} in the original determinant.

Combining the results of this discussion, we see that a determinant may be expanded according to the elements of any column or row by means of the following theorem:

Theorem 1: *The value of a determinant is the sum of the products obtained by multiplying each element of a column (or row) by its minor with a properly prefixed sign. The sign to be used with the element in the* i*th row and* j*th column is* $(-1)^{i+j}$.

Since $(-1)^{i+j}$ is $+1$ or -1 according as $i + j$ is even or odd, we have the following pattern of signs corresponding to the elements of the determinant:

$$\begin{vmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \end{vmatrix}$$

By referring to page 383, it is seen that an expansion by minors according to the elements of the first row has previously been obtained for third-order determinants. As a further example, we write the expansion of a fourth-order determinant according to the elements of the second row:

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = -a_2 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & c_1 & d_1 \\ a_3 & c_3 & d_3 \\ a_4 & c_4 & d_4 \end{vmatrix}$$

$$-c_2 \begin{vmatrix} a_1 & b_1 & d_1 \\ a_3 & b_3 & d_3 \\ a_4 & b_4 & d_4 \end{vmatrix} + d_2 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix}.$$

The expansion may be completed by developing each of the third-order determinants.

An additional property of determinants which will be useful to us is

Property 7: *If, in the expansion of a determinant by minors according to the elements of some one column (row), the elements of this column (row) are replaced by the corresponding elements of any other column (row), the result is zero.*

This property is an immediate consequence of the fact that a determinant with two identical columns has the value zero; for if the replacement described in Property (7) is made, the expansion is that of a determinant with two identical columns. For example, in the expansion

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix},$$

let us replace the a's by the corresponding b's. The result,

$$b_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + b_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix},$$

is the expansion of

$$\begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix} = 0.$$

In evaluating a numerical determinant, we may use the foregoing properties to simplify the work considerably. Some suggested procedures are illustrated in the next examples.

Example 1. Evaluate the determinant

$$D = \begin{vmatrix} 5 & 7 & -1 & 0 \\ 2 & 9 & 5 & 0 \\ 0 & 3 & 8 & -2 \\ 4 & 0 & -3 & 0 \end{vmatrix}.$$

Solution: We note that all but one of the elements of the fourth column are zeros; accordingly we expand by minors along this column. Since a minor multiplied by zero gives zero, it is unnecessary to consider any minor except that of the element -2 in the third row. Therefore,

$$D = -(-2) \begin{vmatrix} 5 & 7 & -1 \\ 2 & 9 & 5 \\ 4 & 0 & -3 \end{vmatrix}.$$

This third-order determinant may now be evaluated by expanding according to the elements of the last row. Thus,

$$D = 2 \left\{ 4 \begin{vmatrix} 7 & -1 \\ 9 & 5 \end{vmatrix} - 3 \begin{vmatrix} 5 & 7 \\ 2 & 9 \end{vmatrix} \right\}$$

$$= 2[4(35 + 9) - 3(45 - 14)] = 166. \quad \textit{Ans.}$$

Example 2. Evaluate the determinant

$$\begin{vmatrix} 6 & 9 & -3 & -3 \\ 2 & 7 & 4 & -5 \\ 1 & 8 & 2 & 3 \\ 4 & 9 & 6 & -2 \end{vmatrix}.$$

Solution: Using Property (4), we factor 3 out of the first row to obtain

$$3\begin{vmatrix} 2 & 3 & -1 & -1 \\ 2 & 7 & 4 & -5 \\ 1 & 8 & 2 & 3 \\ 4 & 9 & 6 & -2 \end{vmatrix}.$$

We next use Property (6) to reduce to zero all except one of the elements of the first row. It is convenient to retain the element -1 in the last position. We add twice each element in the fourth column to the corresponding element in the first column. This procedure gives the result

$$3\begin{vmatrix} 2+2(-1) & 3 & -1 & -1 \\ 2+2(-5) & 7 & 4 & -5 \\ 1+2(3) & 8 & 2 & 3 \\ 4+2(-2) & 9 & 6 & -2 \end{vmatrix} = 3\begin{vmatrix} 0 & 3 & -1 & -1 \\ -8 & 7 & 4 & -5 \\ 7 & 8 & 2 & 3 \\ 0 & 9 & 6 & -2 \end{vmatrix}.$$

Next, add three times each element in the fourth column to the corresponding element in the second column:

$$3\begin{vmatrix} 0 & 3+3(-1) & -1 & -1 \\ -8 & 7+3(-5) & 4 & -5 \\ 7 & 8+3(3) & 2 & 3 \\ 0 & 9+3(-2) & 6 & -2 \end{vmatrix} = 3\begin{vmatrix} 0 & 0 & -1 & -1 \\ -8 & -8 & 4 & -5 \\ 7 & 17 & 2 & 3 \\ 0 & 3 & 6 & -2 \end{vmatrix}.$$

Subtract each element of the fourth column from the corresponding element in the third column:

$$3\begin{vmatrix} 0 & 0 & -1-(-1) & -1 \\ -8 & -8 & 4-(-5) & -5 \\ 7 & 17 & 2-(3) & 3 \\ 0 & 3 & 6-(-2) & -2 \end{vmatrix} = 3\begin{vmatrix} 0 & 0 & 0 & -1 \\ -8 & -8 & 9 & -5 \\ 7 & 17 & -1 & 3 \\ 0 & 3 & 8 & -2 \end{vmatrix}.$$

It is now comparatively simple to complete the expansion of the determinant. First, expand by minors according to the first row to obtain

$$3[-(-1)]\cdot\begin{vmatrix} -8 & -8 & 9 \\ 7 & 17 & -1 \\ 0 & 3 & 8 \end{vmatrix}.$$

Then expand this third-order determinant by minors according to the first column

$$3[-8(136+3) - 7(-64-27)] = -1425. \quad \textit{Ans.}$$

In the preceding problem, we reduced all except one element of the first row to zero by making use of the element -1 in the fourth column. The advantage of having 1 or -1 as an element of the determinant is apparent. If no element in the determinant is 1 or -1, we may obtain such an element by using the same type of transformation as in Example 2. As an illustration, suppose two elements in the same row are 5 and 7, respectively. We may subtract 5 from 7 to obtain 2; then, we may subtract 2 twice from 5 to obtain 1. This procedure carried out on the two columns involved will yield an element 1 in place of the element 5. From this point on, the procedure is as in Example 2. Before attempting to evaluate any determinant, the student should inspect it with care in order to discover an efficient procedure.

EXERCISES 111

Evaluate each of the following determinants:

1. $\begin{vmatrix} 9 & 1 & 7 & 6 \\ 2 & 14 & 2 & 1 \\ -1 & 7 & -1 & 3 \\ 3 & -3 & 2 & 4 \end{vmatrix}$ **2.** $\begin{vmatrix} 7 & 5 & 2 & 4 \\ 1 & 2 & 3 & 1 \\ 3 & -4 & -8 & 3 \\ -5 & -3 & 1 & -2 \end{vmatrix}$

3. $\begin{vmatrix} 2 & 5 & 3 & 0 \\ 1 & 4 & 1 & 3 \\ -4 & 2 & -1 & 3 \\ 3 & 2 & -2 & 1 \end{vmatrix}$ **4.** $\begin{vmatrix} 5 & 3 & 1 & 7 \\ 8 & 6 & 3 & 5 \\ -6 & 2 & 4 & -9 \\ 7 & 5 & 6 & 7 \end{vmatrix}$

5. $\begin{vmatrix} 4 & 6 & 4 & 3 \\ 3 & 1 & -7 & 8 \\ 2 & 3 & -1 & 4 \\ 0 & 2 & -4 & -3 \end{vmatrix}$ **6.** $\begin{vmatrix} 3 & 2 & 1 & 6 \\ 1 & -3 & -6 & -4 \\ 2 & -4 & 2 & 1 \\ -4 & 6 & 19 & 9 \end{vmatrix}$

7. $\begin{vmatrix} -2 & -5 & 3 & -7 \\ 7 & 3 & 8 & -1 \\ 9 & 15 & -10 & 25 \\ 8 & 5 & -3 & 13 \end{vmatrix}$ **8.** $\begin{vmatrix} 6 & -2 & 5 & -1 \\ 2 & 3 & 1 & -2 \\ -7 & 1 & -5 & 3 \\ 4 & 2 & 3 & 4 \end{vmatrix}$

9. $\begin{vmatrix} 0 & 0 & 0 & d \\ 0 & 0 & c & x \\ 0 & b & e & y \\ a & f & g & z \end{vmatrix}$ **10.** $\begin{vmatrix} e & f & h & g \\ h & f & e & g \\ e & g & h & f \\ h & g & e & f \end{vmatrix}$

11. $\begin{vmatrix} 1 & r & s & t \\ a & 1 & 0 & 0 \\ b & 0 & 1 & 0 \\ c & 0 & 0 & 1 \end{vmatrix}$ **12.** $\begin{vmatrix} 0 & -a^2 & a & 1 \\ a^2 & a & 1 & 0 \\ a & 1 & 0 & -a^2 \\ 1 & 0 & a^2 & a \end{vmatrix}$

13. $\begin{vmatrix} x & y & z & 1 \\ 3 & 2 & 6 & 1 \\ 6 & 1 & 3 & 1 \\ 2 & 2 & 5 & 1 \end{vmatrix}$ **14.** $\begin{vmatrix} x^2+y^2 & x & y & 1 \\ 10 & 1 & 3 & 1 \\ 10 & 3 & 1 & 1 \\ 13 & 2 & 3 & 1 \end{vmatrix}$

15. $\begin{vmatrix} 3 & 2 & 3 & 2 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ 1 & -2 & -1 & 0 & 1 \\ -1 & 0 & -3 & -1 & 2 \\ 0 & -1 & 2 & 1 & 1 \end{vmatrix}$ **16.** $\begin{vmatrix} 1 & 4 & -1 & 2 & 3 \\ 2 & 5 & 1 & 3 & 4 \\ 1 & -1 & 2 & -2 & -1 \\ 2 & 1 & 4 & -3 & 1 \\ 3 & -2 & 4 & 1 & 2 \end{vmatrix}$

17. The principal diagonal of a determinant is the diagonal running from the upper left corner to the lower right corner. Show that the value of a determinant is equal to the product of the elements in the principal diagonal if all of the elements lying on one side of that diagonal are zero.

18. If none of the coefficients in the equation

$$a_0x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5 = 0$$

is negative, it may be shown by methods beyond the scope of this book that the equation has only negative real roots or imaginary roots with negative real parts if the following three determinants all have positive values.

$$\begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix}, \quad \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix}, \quad \begin{vmatrix} a_1 & a_0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 \\ a_5 & a_4 & a_3 & a_2 \\ 0 & 0 & a_5 & a_4 \end{vmatrix}.$$

Use this result to show that each of the following equations has only negative real roots or imaginary roots with negative real parts:

(*a*) $z^5 + 3z^4 + 5z^3 + 6z^2 + 2z + 2 = 0$
(*b*) $x^5 + 2x^4 + 7x^3 + 4x^2 + 6x + 1 = 0$

136. The Solution of a System of Linear Equations

The student has undoubtedly surmised that n linear equations in n unknowns can be solved by means of determinants in a manner similar to that used for systems of two and three unknowns. It will be seen from the following discussion that this guess is not very difficult to verify.

For the sake of definiteness, we shall fix our attention on four equations in four unknowns. However, the student should observe that the argument is perfectly general and may be used for n equations in n unknowns.

Let the unknowns be denoted by x_1, x_2, x_3, and x_4, and consider the system

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 &= k_1, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 &= k_2, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 &= k_3, \\ a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 &= k_4. \end{aligned}\right\} \qquad (1)$$

We shall denote by D the determinant of the coefficients on the left sides of the equations in (1); D_1 will denote the determinant obtained from D by replacing the coefficients of x_1 by the corresponding k's; D_2, D_3, and D_4 will denote, similarly, the determinants obtained by replacing the second, third, and fourth columns, respectively, by the k's. We propose to show that, if $D \neq 0$, the solution of the system (1) is given by

$$x_1 = \frac{D_1}{D},\ x_2 = \frac{D_2}{D},\ x_3 = \frac{D_3}{D},\ x_4 = \frac{D_4}{D}.$$

We make use of Property (7), page 393, in order to eliminate all except one of the unknowns from the system. For example, suppose we choose to eliminate all except x_1. Then, we may multiply the first equation by M_{11}, the second equation by $-M_{21}$, the third equation by M_{31}, and the fourth equation by $-M_{41}$. (Recall that M_{ij} stands for the minor of a_{ij}.) Next, we add the four resulting equations to obtain

$$D \cdot x_1 = D_1.$$

This follows from the fact that the sum of the coefficients of x_1 is

$$a_{11}M_{11} - a_{21}M_{21} + a_{31}M_{31} - a_{41}M_{41},$$

which is the expansion of D by minors according to the elements of the first column. The sum of the coefficients of x_2 is the preceding expression with the first-column a's replaced by the second-column a's. Hence, by Property (7), this sum is zero. Similarly, the sum of the coefficients of x_3 and the sum of the coefficients of x_4 are each zero. The sum of the terms in the right members is the preceding expression with the first-column a's replaced by the corresponding k's. This sum is an expansion by minors of the determinant which we agreed to denote by D_1.

By using the minors of the respective a's in the second column and prefixing the correct sign in each equation, we obtain, in a similar manner, the equation

$$D \cdot x_2 = D_2.$$

The corresponding procedures for the third and fourth columns, respectively, yield

$$D \cdot x_3 = D_3,$$

and

$$D \cdot x_4 = D_4.$$

From the method of procedure, it follows that any solution of the given system of equations must also be a solution of the following system:

$$D \cdot x_1 = D_1, \quad D \cdot x_2 = D_2, \quad D \cdot x_3 = D_3, \quad D \cdot x_4 = D_4; \qquad (2)$$

for in the derivation of (2), we have used only the legitimate operations of multiplying both members of an equation by the same constant and of adding equations member by member.

Now, if $D \neq 0$, the solution of the system (2) is obviously

$$x_1 = \frac{D_1}{D}, \quad x_2 = \frac{D_2}{D}, \quad x_3 = \frac{D_3}{D}, \quad x_4 = \frac{D_4}{D}, \quad D \neq 0. \tag{3}$$

We see in this case that the system (1) may have not more than one solution, that given by (3). That this set of values is actually the desired solution may be verified by direct substitution into the system (1).

If $D = 0$ and any one of D_1, D_2, D_3, and D_4 is different from zero, the given system has no solution, and the equations are said to be **inconsistent;** for suppose that $D = 0$ and $D_1 \neq 0$, then, if (x_1', x_2', x_3', x_4') were a solution, we should have from (2)

$$D \cdot x_1' = D_1,$$

that is,

$$0 \cdot x_1' = D_1.$$

But this is impossible if $D_1 \neq 0$.

If $D = 0$ and $D_1 = D_2 = D_3 = D_4 = 0$, the system may or may not have solutions. A complete treatment of this situation is beyond the range of our discussion.

As we noted previously, the methods we have used to obtain the solution of four equations in four unknowns are quite general; the corresponding results may be found by means of the same argument for n equations in n unknowns.

EXAMPLE 1. Use determinants to solve the following system for y:

$$\begin{aligned} x + 2y - 2z + w &= 0, \\ 2x - 3y - 2z - w &= 2, \\ x + 3y + z + w &= 7, \\ 3x + y + z - 2w &= 1. \end{aligned}$$

Solution:

$$D = \begin{vmatrix} 1 & 2 & -2 & 1 \\ 2 & -3 & -2 & -1 \\ 1 & 3 & 1 & 1 \\ 3 & 1 & 1 & -2 \end{vmatrix} = 49;$$

and

$$D_y = \begin{vmatrix} 1 & 0 & -2 & 1 \\ 2 & 2 & -2 & -1 \\ 1 & 7 & 1 & 1 \\ 3 & 1 & 1 & -2 \end{vmatrix} = -98.$$

Thus, $$y = \frac{D_y}{D} = \frac{-98}{49} = -2. \quad \textit{Ans.}$$

EXAMPLE 2. Show that the following system is inconsistent:

$$\begin{aligned} x + 2y - 3z &= 6, \\ 2x - y + z &= 1, \\ 3x + y - 2z &= 8. \end{aligned}$$

Solution: We find

$$D = \begin{vmatrix} 1 & 2 & -3 \\ 2 & -1 & 1 \\ 3 & 1 & -2 \end{vmatrix} = 0,$$

and $$D_x = \begin{vmatrix} 6 & 2 & -3 \\ 1 & -1 & 1 \\ 8 & 1 & -2 \end{vmatrix} = -1 \neq 0.$$

Therefore, the given system is inconsistent.

EXERCISES 112

Use determinants to solve each of the following systems. Do not fail to check your results.

1. $w + 3x - y + 4z = 5$
$2w - x + 2y + z = 4$
$4w + 4x + 3y + 2z = 0$
$3w - 2x + y - z = -5$

2. $r + s + t + u = 6$
$2r + s - t - 3u = 2$
$r - 3s - 2t - u = 2$
$3r - 5s + 2t + 2u = 9$

3. $2r - s + t + u = 2$
$r + 2s + 2t - u = 7$
$r + 5t - 4u = -2$
$3s + 2t - 3u = 0$

4. $w + 2x - 3y - 4z = 1$
$4w + 5x - 2y + 2z = -4$
$w + 3x + 2y - z = 10$
$3w - y + 3z = 2$

5. $a + b + 2c + 3d = 4$
$3a - 4b + 8c + 3d = 8$
$a + 2b - 4c - 3d = -2$
$4a + 5b - 2c + 6d = 7$

6. $2A + 4B + 3C + 3D = 2$
$3A - 3B + C + 2D = 1$
$A + 2B - C + 5D = -2$
$3A + 2C - 4D = 5$

7. $2A + B + C - D = -3$
$A - 2B + C - E = 1$
$3A - B - 3D + 2E = 2$
$A + C - D + E = 6$
$B - C + 2D + E = 6$

8. $3v - 2w + x + y + 2z = 1$
$2v + w + 2x - 3y + z = 0$
$v - w + x - 2y - z = -7$
$v + w + x + y + z = 7$
$v + 4w - 2x + 2y + 3z = 6$

Show that each of the following systems of equations is inconsistent:

9. $2r + s - 3t = 4$
$r - 2s - t = 6$
$4r + 7s - 7t = 12$

10. $3x + 2y + z = 5$
$2x + y - 2z = 4$
$x - 5z = 15$

11. $5u - 3v + 2w = 6$
$4u + 2v - 3w = -10$
$u + 17v - 18w = 7$

12. $6r + 7s - 5t = 0$
$r + s + 3t = 3$
$4r + 5s - 11t = -2$

137. Linear Systems with *m* Equations and *n* Unknowns

There are certain applications of algebra where it is necessary to consider linear systems with the number of equations not equal to the number of unknowns.

We should not generally expect to find that a system with more equations than unknowns has a solution. For in order to have a set of numbers satisfy more conditions than there are numbers, some kind of consistency relation must hold among the conditions. We shall investigate this situation for the case where the number of equations is one more than the number of unknowns.

To be specific, consider the system

$$a_1x + b_1y = c_1,$$

$$a_2x + b_2y = c_2,$$

$$a_3x + b_3y = c_3.$$

We assume that

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0,$$

so that we may solve the first two equations for x and y. In order for the system to be consistent, this solution must satisfy the third equation. The substitution of the values for x and y in the third equation yields

$$a_3 \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} + b_3 \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = c_3.$$

We may multiply both sides by the determinant in the denominator and simplify to obtain

$$a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0.$$

The left member of this equation is the expansion of

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Hence, this third-order determinant must be zero if the system has a solution.

On the other hand, if this determinant is zero, the system may or may not be consistent, as can be seen from a consideration of the three equations

$$x + 2y = 1,$$

$$x + 2y = 2,$$

$$x + 2y = 3.$$

Here, we have

$$\begin{vmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{vmatrix} = 0.$$

In this illustration, however, no two of the three equations are consistent; that is, it is not possible for the same expression $x + 2y$ to have two of the values 1, 2, and 3 simultaneously.

NOTE: For the sake of simplicity, we have assumed that one of the second-order determinants formed from the coefficients of two of the three equations is not zero. It can be shown that the final result is valid without this assumption.

It may be proved in the general case that *if* n *linear equations in* n − 1 *unknowns have a solution, then the* n*th order determinant formed from the array of coefficients and the column of constant terms is zero.* The converse of this statement is not correct, as is seen from the preceding example.

We shall consider next a system that has more unknowns than equations. Usually such a set of equations has an unlimited number of solutions. For example, the system

$$a_1x + b_1y + c_1z = d_1,$$

$$a_2x + b_2y + c_2z = d_2,$$

with

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0,$$

may be written

$$a_1x + b_1y = d_1 - c_1z,$$

$$a_2x + b_2y = d_2 - c_2z.$$

The value of z may now be arbitrarily assigned, and the system will have a solution corresponding to each assigned value of z.

138. Homogeneous Linear Equations

A linear equation in which the constant term is zero is called a **homogeneous** linear equation. Let us consider the following set of n homogeneous linear equations in n unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= 0. \end{aligned}$$

This system always has the so-called *trivial solution*

$$x_1 = 0,\ x_2 = 0,\ \cdots,\ x_n = 0,$$

as can be seen by inspection. It is also clear that, *if the determinant of the coefficients is not zero, there is only the one solution,* since each of the determinants obtained by replacing a column of a's by the constant terms has a column of zeros and is therefore zero.

However, it is shown in higher algebra that *if* $D = 0$, *the system of homogeneous equations always has nontrivial solutions.*

An important instance, which frequently occurs, involves a system of the type

$$\begin{aligned} a_1x + b_1y + c_1z &= 0, \\ a_2x + b_2y + c_2z &= 0. \end{aligned} \tag{1}$$

By introducing a third equation, $0 \cdot x + 0 \cdot y + 0 \cdot z = 0$, we may consider this system of three equations as belonging to the present discussion.

We shall show that

$$x = k\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix},\ y = -k\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix},\ z = k\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

is a solution of Equations (1) for any value of k. By substituting these three values into the left side of the first of Equations (1), we obtain

$$k\left\{a_1\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} - b_1\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + c_1\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\right\}.$$

The expression in braces is the expansion of the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix},$$

which is zero since the first two rows are identical. Hence, the first of Equations (1) is satisfied. It can be shown in the same manner that the second equation is also satisfied.

EXAMPLE 1. Solve the equations

$$2x + 3y + z = 0,$$
$$x - 4y + 3z = 0.$$

Solution:

$$x = k\begin{vmatrix} 3 & 1 \\ -4 & 3 \end{vmatrix} = 13k,$$

$$y = -k\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = -5k,$$

$$z = k\begin{vmatrix} 2 & 3 \\ 1 & -4 \end{vmatrix} = -11k. \quad \textit{Ans.}$$

EXAMPLE 2. Solve the equations

$$x + 2y + 3z = 0, \tag{1}$$
$$5x - y - z = 0, \tag{2}$$
$$3x - 5y - 7z = 0. \tag{3}$$

Solution: Since

$$D = \begin{vmatrix} 1 & 2 & 3 \\ 5 & -1 & -1 \\ 3 & -5 & -7 \end{vmatrix} = 0,$$

the system has nontrivial solutions.

Multiply both members of Equation (1) by 2 and subtract, member by member, from Equation (2) to obtain

$$3x - 5y - 7z = 0.$$

Since this result is the same as Equation (3), we see that any simultaneous solution of (1) and (2) will satisfy (3). Hence, we may follow the method of the preceding example using Equations (1) and (2) only. This gives

$$x = k\begin{vmatrix} 2 & 3 \\ -1 & -1 \end{vmatrix} = k,$$

$$y = -k\begin{vmatrix} 1 & 3 \\ 5 & -1 \end{vmatrix} = 16k,$$

$$z = k\begin{vmatrix} 1 & 2 \\ 5 & -1 \end{vmatrix} = -11k. \quad \textit{Ans.}$$

It can be shown that, if the determinant of the coefficients of three linear homogeneous equations in three unknowns is zero, and if none of the left members is a mere numerical multiple of one of the other left members, the solutions may always be found from any two of the three equations by using the method of Example 1.

EXERCISES 113

For each of the following systems of equations, find the nontrivial solutions if there are any:

1. $2x + y - 4z = 0$
$x - 3y + z = 0$

2. $3A + 4B - C = 0$
$A - 2B - C = 0$

3. $3u - 2v + 4w = 0$
$2u + 5v - 3w = 0$

4. $r + 5s - 2t = 0$
$4r - s + 7t = 0$

5. $2x + y - 3z = 0$
$x - 2y - z = 0$
$4x + 7y - 7z = 0$

6. $6u + 7v - 5w = 0$
$u + v + 3w = 0$
$4u + 5v - 11w = 0$

7. $5A - 2B + 7C = 0$
$3A + 4B + 6C = 0$
$26B + 9C = 0$

8. $x - 3y + 4z = 0$
$2x - 5y + 2z = 0$
$5x + y - 3z = 0$

9. $2r - 3s - 8t = 0$
$5r - 2s - 3t = 0$
$3r + 3s + 11t = 0$

10. $4A - 5B - 2C = 0$
$6A - 7B - 3C = 0$
$2A - 4B - C = 0$

In each of the following examples, find the value, or values, of m for which the given system has nontrivial solutions. Then, find these solutions.

11. $2mA - 3B - 2C = 0$
$3mA + B + 2C = 0$
$3A + 7B + 6C = 0$

12. $3r - 7s + 2t = 0$
$mr - 2ms + 3t = 0$
$7r - 11s + 6t = 0$

13. $8u + 5v - 5w = 0$
$2mu + 3v - w = 0$
$2u + v - mw = 0$

14. $x - 4y + 13z = 0$
$mx + 2y - z = 0$
$2x - y + 2mz = 0$

Find solutions, when they exist, for each of the following systems of equations:

15. $2x + y = 1$
$3x + 2y = -1$
$7x + 3y = 6$

16. $u + 4v = 18$
$3u - v = 2$
$2u + 3v = 16$

17. $x + 3y = 1$
$2x - y = 5$
$4x + 5y = 4$

18. $2r + 4s = -7$
$3r - 5s = 17$
$5r - 3s = 15$

19. $8x + 4y = 1$
$3x + 5y = 3$
$5x + 3y = 1$

20. $5A + 2B = 4$
$2A + 4B = 5$
$3A + B = 2$

21. $3x + y - z = 4$
$x - 2y + 9z = 6$

22. $4r - 3s + 4t = -3$
$r + 2s - 10t = 13$

23. $10A + 3B - C = 1$
$12A + 5B + 3C = 4$

24. $2x - 9y + 14z = -5$
$4x + 3y - 7z = 11$

Chapter 20

PERMUTATIONS, COMBINATIONS, AND PROBABILITY

The topics of permutations and combinations find one of their most important applications in the subject of probability. The theory of probability is itself fundamental in the study of statistics, which in turn is the basis for important branches of the natural sciences, as well as being the foundation of the modern business of life insurance.

Furthermore, many questions both interesting and instructive may be answered by application of the fundamental ideas of permutations, combinations, and probability.

Although it is impossible to give an extensive treatment of these topics in a few pages, some of the elementary aspects will be discussed.

139. The Fundamental Principle

We shall illustrate the reasoning on which the work of this chapter is based by a simple example. Suppose that in going from city A to city C, it is necessary to pass through city B. Suppose further that it is possible to go from A to B by four different routes and from B to C by three different routes. Then, the total number of different routes from A to C is $3 \cdot 4 = 12$. For, we may go from A to B in any one of four different ways, and for each of these ways there are three ways of going from B to C.

Since this reasoning may be applied in general, we state it in the following form:

Fundamental Principle: *If an act may be performed in any one of* m *different ways, and if, after it is done, a second act may be performed in any*

one of n *different ways, then the total number of ways in which the two acts may be performed in the stated order is* mn.

Clearly, the same reasoning may be extended to include any number of successive acts.

EXAMPLE 1. In how many ways can five boys and four girls form boy and girl couples?

Solution: Any one of the five boys may be chosen for the first member of a couple, and, after this is done, any one of the four girls may be chosen for the boy's partner. Hence, by the fundamental principle, the total number of different couples possible is

$$5 \cdot 4 = 20. \quad Ans.$$

EXAMPLE 2. How many three-digit integers less than 500 can be formed from the digits 1, 3, 5, and 7 if repetition of digits is not allowed?

Solution: The hundreds' place may be filled in either one of two ways, namely, by the digits 1 or 3, as the number is to be less than 500. The tens' place may then be filled in any one of three ways, namely, by any one of the three numbers that are left. And, after making a selection for the tens' place, the units' place may be filled in any one of two ways. Hence, the total number of different ways in which all three places can be filled is

$$2 \cdot 3 \cdot 2 = 12. \quad Ans.$$

EXAMPLE 3. For the data in Example 2, how many numbers can be formed if repetition of digits is allowed?

Solution; As before, the hundreds' place may be filled in any one of two different ways; however, with repetitions allowed, the tens' and the units' places may each be filled in any one of four different ways. Hence, the total number of different ways in which all three places can be filled is

$$2 \cdot 4 \cdot 4 = 32. \quad Ans.$$

EXERCISES 114

1. Three clubs have twelve, fourteen, and fifteen members, respectively. How many committees of three people each can be picked if all three organizations are to be represented on each committee?

2. From the top of a mountain there are four ski runs to a warming house, and from this shelter there are seven more runs to the bottom of the mountain. In how many ways is it possible to ski the entire length of the mountain?

3. A person purchasing an automobile from a certain company has a choice of four body styles, a choice of two motors, and a choice of 10 colors. In how many ways is it possible to choose a car?

4. Nine persons apply for three different positions. In how many ways can these positions be filled?

5. If there are twelve entries in a contest, in how many ways can first, second, and third prizes be awarded?

6. In how many ways can a person enter a dormitory and leave by a different door if there are eight doors?

7. In how many ways can three persons enter a stadium that has ten gates if no two of the persons choose the same gate?

8. In throwing three dice, in how many ways can they fall? (Consider each die separately.)

9. In how many ways can three persons enter a stadium that has ten gates?

10. There are eleven boys and eight girls in a tennis club. In how many ways can a doubles match be arranged if each side is composed of a boy and a girl?

11. How many different integers are exactly divisible into $3^3 \cdot 5^2 \cdot 7$?

12. How many different integers are exactly divisible into $2^4 \cdot 3^2 \cdot 5^3 \cdot 7^2$?

140. Permutations

Each arrangement of a set of things, or of a part of the set, in some order in a straight line is termed a **permutation** of the set. For example, the set of digits 7, 8, 9 may be arranged in the following six ways, each of which is a permutation of the set:

789	879	978
798	897	987

The number of permutations may be calculated beforehand by noting that the first place can be filled in any one of three different ways; after it is filled, the second place can be filled in any one of two different ways; and, then, the third place is filled by the remaining digit. Therefore, in accordance with our fundamental principle, there are

$$3 \cdot 2 \cdot 1 = 3! = 6$$

ways of permuting the three digits.

If we have *four* letters x, y, z, and w, we may have the following twelve arrangements of them taken *two* at a time:

xy	yx	zx	wx
xz	yz	zy	wy
xw	yw	zw	wz

Again, the number of possible permutations can be found beforehand. This is accomplished by observing that the first place may be filled in any one of four different ways, and, after it is filled, the second place may be filled in three different ways. Therefore, there are

$$4 \cdot 3 = 12$$

ways of permuting the four letters taken two at a time.

We may arrive at the corresponding result in the general case by the same argument as we used in the preceding illustrations. If we have n different objects to be permuted k at a time, the first position may clearly be filled in n different ways; after making the first selection, the second position may be filled in $n - 1$ different ways, and so on; the kth position can be filled in $n - k + 1$ different ways. Hence, denoting the number of permutations of n different things, k at a time, by $P(n, k)$, we have

$$P(n, k) = n(n - 1)(n - 2) \cdots (n - k + 1), \tag{1}$$

there being k factors in the product.

In particular, if $k = n$, we obtain the number of permutations of n different things, n at a time, as

$$P(n, n) = n(n - 1)(n - 2) \cdots (2)(1) = n! \tag{2}$$

It should be emphasized that the formulas given by (1) and (2) apply only when the objects to be permuted are all *different.*

Example 1. How many three-digit integers can be written with the digits 1, 2, 3, 4, 5 if no repetition of digits is allowed?

Solution: We have five different digits to be taken three at a time. Therefore, the required number is

$$P(5, 3) = 5 \cdot 4 \cdot 3 = 60. \quad \textit{Ans.}$$

Example 2. How many four-digit integers can be written with the digits 0, 1, 2, 3, 4, 5 if no repetitions are allowed?

Solution: Since the thousands' place may not be filled with 0, there are only five different digits for this position. After the thousands' place is filled, there are five digits left for the other three places. Hence, the total number of numbers satisfying the conditions is

$$5 \cdot P(5, 3) = 5(5 \cdot 4 \cdot 3) = 300. \quad \textit{Ans.}$$

We consider next the permutations of n things not all different, taking them n at a time. For example, to find the number of possible permutations of the letters in the word ELEVEN, using all the letters, we

first mark the three e's so that they become distinguishable, say e_1, e_2, e_3. The six different symbols e_1, l, e_2, v, e_3, and n may be arranged in

$$P(6, 6) = 6!$$

different ways; and one of these 6! permutations is

$$e_1le_2ve_3n.$$

If the l, v, and n are held in these positions, the three e's may be permuted in

$$P(3, 3) = 3!$$

ways. However, if the subscripts attached to the e's were erased, these 3! permutations would be indistinguishable from each other. This means that to each distinct permutation with the e's alike, there correspond 3! permutations with the e's different. Therefore, if P is the number of distinct arrangements, the total number of arrangements with different e's is $3!P$. But this total number is $P(6, 6)$. Hence,

$$3!P = 6!$$

or

$$P = \frac{6!}{3!} = 120.$$

In dealing with a set of n things, some of which are duplicates of others, we may separate the like things into groups such that no group contains the duplicate of a member of another group. By applying the reasoning of the preceding paragraph, we find that the number of permutations of n things taken all at a time, of which n_1 are alike, n_2 others are alike, and so on, is

$$P = \frac{n!}{n_1!n_2!n_3! \cdots}. \tag{3}$$

EXAMPLE 3. How many distinct permutations are there of all the letters in the word POTATO?

Solution: We have six letters with two o's and two t's; hence, Formula (3) gives

$$P = \frac{6!}{(2!)(2!)} = 180. \quad Ans.$$

It is sometimes necessary to consider the arrangement of objects in a circle rather than in a straight line. In this case, we agree that two arrangements are the same if one can be obtained from the other by rotation alone. Hence, if there are n objects, and any one of them is

placed, the remaining ones may be permuted in $(n-1)!$ ways. Thus, the number of permutations of n different objects, n at a time, in a circle is $(n-1)!$.

EXERCISES 115

1. Find n, if $P(n, 2) = 110$.

2. If $P(n+1, 2) = 3P(n-3, 2)$, find n.

3. Find n, if $7P(n, 2) = 12P(n-2, 2)$.

4. A president, a vice-president, a secretary, and a treasurer of an organization are selected from thirty members. In how many ways can this selection be made?

5. There are twelve men available for the four backfield positions of a football team. If we assume that each of the twelve men can fill any position, in how many ways can these positions be filled?

6. How many three digit numbers can be formed from the digits 1, 3, 4, 7, 8, 9 if no repetitions of digits are allowed?

7. How many permutations can be made of the letters in the word CONSIDER if four letters are taken at a time? if six letters are taken at a time?

8. In how many ways can eight people be seated at a round table?

9. In how many ways can twelve people be seated at a round table if two particular people sit opposite one another?

10. In how many ways can five men and five women be seated at a round table so that the men and women are seated alternately?

11. How many odd numbers, each consisting of four digits, can be formed from the digits 1, 2, 4, 5, 6, 8, 9 with no repetitions allowed?

12. How many even numbers, each consisting of four digits, can be formed from the digits 1, 2, 4, 5, 6, 8, 9 with no repetitions allowed?

13. How many numbers of not more than four digits each can be formed from the digits 3, 4, 5, 6 if repetitions are not allowed?

14. How many numbers of not more than four digits each can be formed from the digits 3, 4, 5, 6 if repetitions are allowed?

15. How many six-digit numbers can be formed from the digits 2, 4, 5, 6, 8, 9 so that the digits 4 and 8 are not adjacent to one another? (HINT: First, consider the number of nonpermissible arrangements.)

16. In how many different orders of distinctly different appearance can a row of five pennies, four nickels, four dimes, and three quarters be arranged?

17. There are eleven flags that are displayed together, one above another, on a flagpole. How many signals are possible if four of the flags are blue, two are red, three are yellow, and two are white?

18. How many distinct permutations can be made of the letters in the word SPEEDERS, taken all at a time? How many of these will have P in the first place?

19. How many distinct permutations can be made from the ten letters in the word DIFFERENCE, taken all at a time?

20. How many distinct permutations of seven letters each can be formed from the word WITHOUT if the vowels are to occupy the even places?

21. At a football game there are three groups of people sitting in a row. There are eight persons in the first group, seven in the second group, and four in the third group. In how many ways can these people be arranged in the row if all the persons of the same group must be kept together?

22. In how many ways can five different books on mathematics, four different books on history, and three different books on psychology be arranged on a shelf so that those on the same subject shall be together? How many different arrangements are possible if only the books on history must be kept together?

23. In how many ways can ten different flags be arranged consecutively on a pole so that three particular flags shall not all be together?

24. In how many ways can ten different flags be arranged consecutively on a pole so that no flag of a particular group of three flags shall be next to any other flag of that group?

141. Combinations

A set of k different things selected *without regard to order* from a set of n different things is called a **combination of n things, taken k at a time.** Thus, xyz is one of the combinations of the four letters x, y, z, w taken three at a time. Furthermore, xyz and zyx are the same combination, although they are different permutations. Clearly, then, there are more permutations than combinations of n things k at a time.

Since a set of k different things may be permuted among themselves in $k!$ different ways, each combination of n things k at a time must furnish $k!$ of the total number of permutations of the n things k at a time. Therefore, if the number of combinations is denoted by $C(n, k)$, we have

$$k! \cdot C(n, k) = P(n, k),$$

$$C(n, k) = \frac{P(n, k)}{k!}$$

$$= \frac{n(n-1) \cdots (n-k+1)}{k!}. \tag{4}$$

EXAMPLE 1. How many different subcommittees of three persons each can be chosen from a committee of ten persons?

Solution: Since the order in which the members of a subcommittee are chosen is assumed to be immaterial, we must find the number of

combinations of ten things taken three at a time. Hence, the required number is

$$C(10, 3) = \frac{10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3} = 120. \quad Ans.$$

Each time that a combination of k things is taken from a set of n things, a combination of $n - k$ things is left. From this statement, it follows that

$$C(n, k) = C(n, n - k). \tag{5}$$

We may verify this fact directly by noting that

$$C(n, k) = \frac{n(n-1)\cdots(n-k+1)}{k!} \cdot \frac{(n-k)!}{(n-k)!} = \frac{n!}{(n-k)!k!}$$

and $$C(n, n-k) = \frac{n(n-1)\cdots(k+1)}{(n-k)!} \cdot \frac{k!}{k!} = \frac{n!}{(n-k)!k!}.$$

Thus, *the number of combinations of* n *things* k *at a time is the same as the number of combinations of* n *things* n − k *at a time.*

Example 2. Find the number of combinations of twenty things taken seventeen at a time.

Solution: Using Formula (5), we have

$$C(20, 17) = C(20, 3) = \frac{20 \cdot 19 \cdot 18}{1 \cdot 2 \cdot 3} = 1140. \quad Ans.$$

Note that it is much more efficient to develop $C(20, 3)$ than $C(20, 17)$.

In problems involving permutations and combinations, we must be careful to differentiate between situations according as the arrangement of things is, or is not, to be considered. If the arrangement is to be taken into account, the problem involves permutations; if the arrangement is to be disregarded, the problem involves combinations. It is advisable always to rely on fundamental ideas rather than on formulas; each problem should be carefully analyzed, and a formula should be used only when it fits the situation exactly.

Example 3. In a certain county, there is to be appointed a commission consisting of two bankers and three engineers. If five bankers and six engineers are candidates for the appointments, how many different committees are there from which to choose?

Solution: From the five bankers, two may be chosen in any one of

$C(5, 2)$ ways. Similarly, from the six engineers, three may be selected in any one of $C(6, 3)$ ways. Since any choice of the bankers may be combined with any choice of the engineers, the required number is

$$C(5, 2) \cdot C(6, 3) = \frac{5 \cdot 4}{1 \cdot 2} \cdot \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} = 200. \quad \textit{Ans.}$$

142. Combinations and the Binomial Formula

By comparison of the formula

$$C(n, k) = \frac{n(n-1) \cdots (n-k+1)}{k!}$$

with the coefficients in the binomial formula

$$(a+b)^n = a^n + \frac{n}{1} a^{n-1}b + \frac{n(n-1)}{1 \cdot 2} a^{n-2}b^2 + \cdots + \frac{n(n-1) \cdots (n-k+1)}{k!} a^{n-k}b^k + \cdots + b^n,$$

we see that $C(n, k)$ is the coefficient of the $(k+1)$th term in the expansion of $(a+b)^n$, where n is an integer. Hence, we may write

$$(a+b)^n = a^n + C(n, 1)a^{n-1}b + C(n, 2)a^{n-2}b^2 + \cdots + C(n, k)a^{n-k}b^k + \cdots + C(n, n)b^n.$$

In fact, we may derive the binomial formula directly by means of combinations as follows:

$$(a+b)^n = (a+b)(a+b) \cdots (a+b), \text{ for } n \text{ factors.}$$

Hence, each term of the final expansion may be regarded as the product of n letters, one taken from each factor. For example, a product $a^{n-k}b^k$ would be obtained by taking a from $n-k$ factors and b from the remaining k factors. It is evident that the number of ways in which such a product can be formed is $C(n, k)$, the number of ways in which k things can be selected from n things.

143. The Total Number of Combinations

If in the expansion of $(a+b)^n$, we set $a = b = 1$, we get

$$(1+1)^n = 2^n = 1 + C(n, 1) + C(n, 2) + \cdots + C(n, n),$$

and, by subtracting 1 from both members, we find

$$C(n, 1) + C(n, 2) + \cdots + C(n, n) = 2^n - 1. \tag{6}$$

This formula gives *the total number of combinations of* n *things taken* 1, 2, 3, $\cdots$, n *at a time.*

Example 1. Verify the preceding formula for $n = 3$ by finding the total number of combinations of the letters x, y, and z, taken one, two, or three at a time.

Solution:
$$C(3, 1) = \frac{3}{1} = 3,$$
$$C(3, 2) = \frac{3 \cdot 2}{1 \cdot 2} = 3,$$
and
$$C(3, 3) = \frac{3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3} = 1.$$

So, the total number of combinations is seven; also, if $n = 3$, $2^n - 1 = 7$. The seven combinations may be written out as follows:

x	y	z	(1 at a time)
xy	yz	zx	(2 at a time)
	xyz		(3 at a time)

EXERCISES 116

1. Evaluate (*a*) $C(28, 24)$; (*b*) $C(41, 38)$.

2. If $P(n, r) = 840$ and $C(n, r) = 35$, find n and r.

3. A sports dealer has ten new fishing rods to display. If he wishes to display only three rods at a time, in how many ways can he choose them?

4. In how many ways can five committeemen be selected from a group of eleven men?

5. A ski club has fifteen men who desire to be on a five-man ski team. Find the number of different ski teams that can be formed.

6. If twenty-eight people all shake hands with one another, how many handshakes are there?

7. How many five-man basketball teams can be formed from a squad of sixteen players if two men can play the center position only, and each of the remaining men can play in any other position?

8. There are eleven points, no three of which are in the same straight line. How many lines are determined by these points?

9. From a squad of twenty men, three men can catch and three men can pitch. If these six men can play at no other positions, how many baseball nines can be formed?

10. How many rectangles are formed by seven parallel lines intersecting another set of nine parallel lines which are perpendicular to the first set?

11. How many four-letter permutations can be formed from the alphabet if each permutation contains the letters a and e? No repetitions are allowed.

12. How many five-letter permutations can be formed from the alphabet if each permutation contains a, e, and o? No repetitions are allowed.

13. A department store wishes to fill ten positions with four men and six women. In how many ways can these positions be filled if nine men and eleven women apply for them?

14. In how many ways can a person choose five cards from a pack of fifty-two cards if three of them are to be clubs?

15. From a group of eighteen men a committee of five is to be selected. How many different committees can be formed? On how many different committees can some one particular man serve?

16. Find the total number of different weights that can be formed with five objects that weigh 1, 3, 5, 10, and 20 lb, respectively.

17. What is the total number of combinations that can be made with the letters a, b, c, d, e, f, g, h?

18. How many different sums of money can be formed from a penny, a nickel, a dime, a quarter, a half dollar, and a dollar?

19. How many different sums of money can be formed from the coins in the preceding exercise if each sum consists of two or more coins?

20. How many committees can be formed from twelve men if each committee must consist of at least three men?

21. How many diagonals does a polygon of eight sides have? of n sides? (A diagonal is a line joining two nonadjacent vertices.)

22. A committee of five men is to be chosen from among eleven men. If a certain two men cannot serve on the same committee, how many committees can be formed?

23. A committee of five men is to be chosen from among eleven men. If a certain two men must serve on the same committee, but do not have to serve on all committees, how many committees can be formed?

24. In how many ways can ten objects be divided into two equal groups?

25. In how many ways can ten objects be divided into two groups?

26. A pack of fifty-two playing cards is divided into four hands of thirteen cards each. If the order of the hands is taken into account, in how many ways can this be done?

27. In how many ways can ten different planes be distributed among five airfields so that each field receives two planes?

144. Probability

If a nickel and a dime are tossed, they may fall in any one of four different ways: both heads up; both tails up; nickel tails up, and dime heads up; or, nickel heads up, and dime tails up. So far as we know, any one of these falls is as likely to occur as any other. We say, there-

fore, that the probability of two heads is $\frac{1}{4}$, the probability of two tails is $\frac{1}{4}$, and the probability of one head and one tail is $\frac{2}{4}$ or $\frac{1}{2}$.

The mathematical measure of probability is defined in the following way:

Mathematical Probability: *If on any trial, an event can happen in any one of* m *ways and can fail to happen in any one of* n *ways, and if these* m + n *ways are all equally likely, then the probability of the event happening is*

$$\mathrm{p} = \frac{\mathrm{m}}{\mathrm{m} + \mathrm{n}},$$

and the probability of its failing to happen is

$$\mathrm{q} = \frac{\mathrm{n}}{\mathrm{m} + \mathrm{n}}.$$

For example, when two coins are tossed, there are four ways in which the coins may fall. In three of these ways, at least one head occurs; and, in the remaining one way, heads fail to occur. Hence, the probability that at least one head will turn up is

$$p = \frac{3}{3 + 1} = \frac{3}{4}.$$

The probability that heads fail to occur is

$$q = \frac{1}{3 + 1} = \frac{1}{4}.$$

If follows from the definition that, if an event is certain to happen, its probability of happening is 1; and, if it is certain not to happen, its probability is 0. If it is certain that an event will either happen or fail to happen, the sum of the probabilities of success and failure is 1.

In many important instances it is impossible to make an analysis of an event so that the number of equally likely ways in which it can happen or fail to happen is known. However, it may be possible to observe a large number of trials of the event and to record the number of successes and failures. The ratio of the number of successes to the total number of trials is then called the **relative frequency** of occurrence of the event. If the number of trials is quite large, we assume that the relative frequency is approximately equal to the probability and may be used in place of it. To distinguish this type of probability from mathematical probability, the name **empirical probability** is sometimes used.

For example, suppose an inspector examines a sample lot of 1000 small machine parts and finds 25 defective pieces. On the basis of this experiment, we should say that the empirical probability of a part being defective is $\frac{25}{1000}$ or 0.025. In this situation, there is no practical possibility of determining the mathematical probability.

As another important example of empirical probability, it may be noted that until recently insurance companies based their computations on the *American Experience Table of Mortality.* This table gives the number of persons surviving at various ages out of an initial group of 100,000 alive at ten years of age. The following items are taken from this table:

MORTALITY TABLE

Age	Number Living	Age	Number Living
10	100,000	60	57,917
20	92,637	70	38,569
30	85,441	80	14,474
40	78,106	90	847
50	69,804	100	0

We see that, of the group of 100,000 persons alive at ten years of age, 69,804 were still alive at age fifty. Hence, the probability of a child of age ten living to age fifty is taken as

$$\frac{69{,}804}{100{,}000} = 0.698 \text{ approximately.}$$

EXAMPLE 1. Calculate from the preceding table the probability that a person of age twenty will die before age thirty.

Solution: We read from the table that of 92,637 persons alive at age twenty, 85,441 are still alive at age thirty. Therefore, the number dying between ages twenty and thirty is

$$92{,}637 - 85{,}441 = 7{,}196,$$

and the required probability is

$$\frac{7{,}196}{92{,}637} = 0.0777 \text{ approximately.} \quad \textit{Ans.}$$

The remaining fundamental ideas in our discussion of probability will be dealt with in connection with examples.

EXAMPLE 2. Find the probability of throwing an 11 in one trial with a pair of dice.

Solution: Each die can fall in six different ways; so that a pair of dice can fall in $6 \cdot 6 = 36$ ways. Of these ways, the sum 11 is obtainable in only two ways: 6 and 5 or 5 and 6. Hence, the required probability is $\frac{2}{36}$ or $\frac{1}{18}$. *Ans.*

We may also reason as follows: One of the dice can turn up a 5 or a 6 in two ways, and the probability that this event will happen is $\frac{2}{6}$. If one of these numbers comes up, the second die can fall in only one way to give the sum 11. The probability that the second die falls favorably is, therefore, $\frac{1}{6}$. Consequently, the probability that an 11 will be thrown is

$$\tfrac{2}{6} \cdot \tfrac{1}{6} = \tfrac{1}{18}, \text{ as before.}$$

The reasoning in Example 2 illustrates the general principle: *If the probabilities of the occurrence of* n *independent events are* $p_1, p_2, \cdots, p_n$, *respectively, then the probability that all the events will occur is*

$$p = p_1 \cdot p_2 \cdot \cdots \cdot p_n.$$

EXAMPLE 3. From a box containing five black balls and three white balls, 3 balls are drawn in succession. Find the probability that all are white (*a*) if each ball is returned to the box before the next is drawn; (*b*) if the balls are not returned to the box.

Solution: (*a*) The probability of getting a white ball on any drawing is $\frac{3}{8}$; and each of the three drawings is independent of the other two. Hence, the required probability is

$$\tfrac{3}{8} \cdot \tfrac{3}{8} \cdot \tfrac{3}{8} = \tfrac{27}{512} = 0.0527-. \quad \textit{Ans.}$$

(*b*) The probability of getting a white ball the first time is $\frac{3}{8}$. If a white ball is drawn, the probability of getting a white ball the next time is $\frac{2}{7}$; and if the second white ball is drawn, the probability of drawing a white ball on the third trial is $\frac{1}{6}$. Therefore, the required probability is

$$\tfrac{3}{8} \cdot \tfrac{2}{7} \cdot \tfrac{1}{6} = \tfrac{1}{56} = 0.0179-. \quad \textit{Ans.}$$

The next example deals with mutually exclusive, rather than independent, events.

EXAMPLE 4. Four coins are tossed. What is the probability that (*a*) all four will turn up alike? (*b*) at least two will be heads?

Solution: (*a*) The total number of ways in which four coins can fall is $2^4 = 16$. Of these ways, only two are favorable, that is, all heads or

all tails. Hence, the required probability is

$$\tfrac{2}{16} = \tfrac{1}{8}. \quad \textit{Ans.}$$

Also, the probability of getting all heads is $\frac{1}{16}$, and the probability of getting all tails is $\frac{1}{16}$. Notice that the answer to part (a) is $\frac{1}{16} + \frac{1}{16} = \frac{1}{8}$.

(b) In order to calculate the probability that there will be at least two heads, we may use the following analysis: There is one way of getting all heads; and there are $C(4, 3)$ ways of getting three heads and $C(4, 2)$ ways of getting two heads. Hence, the number of ways of having at least two heads is

$$1 + C(4, 3) + C(4, 2) = 1 + 4 + 6 = 11.$$

Thus, the required probability is $\frac{11}{16}$. *Ans.*

In part (b), the probability of all heads is $\frac{1}{16}$, of three heads is $\dfrac{C(4, 3)}{16}$, and of two heads is $\dfrac{C(4, 2)}{16}$. As in part (a), our answer is again obtained by adding these three probabilities.

The argument in Example 4 may be used in general to demonstrate the principle: *If the probabilities of the occurrence of* n *mutually exclusive events are* $p_1, p_2, \cdots, p_n$, *respectively, the probability that one of these events will occur is*

$$p = p_1 + p_2 + \cdots + p_n.$$

Example 5. A box contains five white balls and three black balls. Find the probability that, in four successive drawings, (a) exactly one white ball will be obtained; (b) exactly two white balls will be obtained; (c) that either one or two white balls will be obtained. The balls are returned to the box after each trial.

Solution: (a) Suppose that the white ball is to be obtained in the first drawing; the probability of this occurrence is $\frac{5}{8}$. Also, the probability that a black ball is drawn in any trial is $\frac{3}{8}$. Therefore, the probability that one white and three black balls are drawn in the stated order is $(\frac{5}{8})(\frac{3}{8})(\frac{3}{8})(\frac{3}{8}) = (\frac{5}{8})(\frac{3}{8})^3$. Evidently, this result is the probability of getting exactly one white ball regardless of the order in which the colors are assumed to be drawn. Since there are $C(4, 1) = 4$ different positions in which the white ball may be drawn, the required probability is

$$C(4, 1)\left(\frac{5}{8}\right)\left(\frac{3}{8}\right)^3 = \frac{135}{1024} = 0.132-. \quad \textit{Ans.}$$

(*b*) As in part (*a*), we see that the probability of drawing exactly two white balls in any chosen position in four drawings is $(\frac{5}{8})(\frac{5}{8})(\frac{3}{8})(\frac{3}{8})$ or $(\frac{5}{8})^2(\frac{3}{8})^2$. Furthermore, the number of ways in which the positions of the white balls can be chosen is $C(4, 2) = 6$. Any of these positions is permissible; therefore, the probability that exactly two white balls will occur is

$$C(4, 2)\left(\frac{5}{8}\right)^2\left(\frac{3}{8}\right)^2 = \frac{675}{2048} = 0.329+. \quad \textit{Ans.}$$

(*c*) Since the occurrence of exactly one white ball and the occurrence of exactly two white balls are mutually exclusive events, the probability that either one or two white balls occur is the sum of the answers to parts (*a*) and (*b*):

$$\frac{135}{1024} + \frac{675}{2048} = \frac{945}{2048} = 0.461+. \quad \textit{Ans.}$$

EXERCISES 117

1. Five judges for a certain contest are to be picked at random from a group of 50 people. What is the probability that a certain person in the group will be chosen?

2. Twelve persons are paired at random to play in a singles tennis tournament. What is the probability that two particular players will be paired with one another?

3. Eleven persons are to be seated at a round table. What is the probability that two particular persons will sit together?

4. A small coin bank contains seven dimes and nine quarters. What is the probability of shaking two dimes in succession out of the bank?

5. A committee of five is to be chosen at random from ten boys and eight girls. What is the probability there will be three boys and two girls on the committee?

6. A committee of seven is to be chosen at random from twelve men and eleven women. What is the probability that there will be four men and three women on the committee?

7. Find the probability of throwing a 9 in a single throw of two dice.

8. Two cards are drawn from a suit of thirteen cards. Find the probability that the ten and jack are drawn.

9. What is the probability of throwing at least a 4 in two throws of a die?

10. If five coins are tossed, what is the probability that exactly three of them are heads?

11. In throwing a die five times what is the probability that an ace will turn up exactly two times?

12. A baseball player has a batting average of 0.300. Find the probability that he will get exactly three hits out of four times at bat.

13. In a rifle club, A is able to hit the bull's-eye 9 times out of 10; B, 85 times out of 100; and C, 95 times out of 100. What is the probability of their hitting the bull's-eye at least once if they shoot simultaneously?

14. If thirteen coins are tossed, what is the probability that at least five of them will be tails?

15. If ten coins are tossed, what is the probability that at least six of them will be heads?

16. What is a person's chance of throwing an 8 or higher in a single throw of two dice?

17. What is the probability that of four cards drawn from a deck of fifty-two cards three will be hearts and one will be a diamond?

18. From a box that contains eight red balls and six white balls, five balls are taken at random. What is the probability that all the balls are red? What is the probability that at least three are red?

19. A person draws three cards from a suit of thirteen cards. What is the probability that a face card is not drawn? What is the probability that one or more face cards are drawn? What is the probability of drawing all face cards?

20. If four blue books, six green books, and five red books are placed at random on a shelf, what is the probability that the blue books will all be together?

21. If four blue books, six green books, and five red books are placed at random on a shelf, what is the probability that at least three green books will be together?

22. Find the probability that a man forty years old will be dead twenty years later. (Refer to the mortality table on page 418.)

23. Find the probability that a child ten years old will die in the next thirty years. Also find the probability that he will live for at least thirty years.

24. What is the probability that a person twenty years old will live to be thirty and die before he reaches sixty?

25. What is the probability that five persons thirty years old will all be alive ten years hence? At least three will be alive?

Chapter 21

PARTIAL FRACTIONS

145. Introduction

In the more elementary portions of algebra, the student learns how to combine a group of simple algebraic fractions joined by plus and minus signs into a single fraction. It is easy to show, for instance, that

$$\frac{2}{x-1} - \frac{1}{x+1} = \frac{x+3}{x^2-1}.$$

The reverse process of decomposing a fraction into a sum of simple fractions is frequently of importance in more advanced work, notably in the calculus. Each of the simpler fractions occurring in this process is called a **partial fraction,** and the process itself is called **decomposition into partial fractions.** Thus, in the preceding illustration, the two fractions on the left, $2/(x - 1)$ and $-1/(x + 1)$, are the partial fractions corresponding to the fraction $(x + 3)/(x^2 - 1)$.

We shall be concerned entirely with **rational** fractions having real coefficients, that is, with fractions of the type

$$\frac{a_0x^m + a_1x^{m-1} + \cdots + a_m}{b_0x^n + b_1x^{n-1} + \cdots + b_n},$$

where the a's and the b's are real constants, n is a positive integer, and m is a positive integer or zero. For the sake of brevity, we shall denote a rational fraction by $p(x)/q(x)$ with the understanding that $p(x)$ and $q(x)$ stand identically for the polynomials in the numerator and in the denominator, respectively.

A **proper** fraction is one whose numerator is of lower degree than its

denominator. If $p(x)$ is of degree greater than or equal to that of $q(x)$, we can divide $p(x)$ by $q(x)$ and obtain an integral expression plus a proper fraction. For example,

$$\frac{x^4 + 2x^3}{x^2 - 1} = x^2 + 2x + 1 + \frac{2x + 1}{x^2 - 1}.$$

We shall assume in the remainder of this chapter that such a division, if possible, is always performed before any other decomposition is attempted. Accordingly, our work will concern itself entirely with the decomposition of proper rational fractions in their lowest terms.

Since every polynomial with real coefficients has only conjugate imaginary zeros and/or real zeros, it follows that the polynomial $q(x)$ has only real quadratic and/or real linear factors. As will appear presently, the type of partial fraction we use depends upon the factorization of $q(x)$.

146. Simple Linear Factors

A factor of the polynomial $q(x)$ is called **simple** if it occurs to the first and no higher power. If $x - r$ is a simple factor of $q(x)$, then $q(x) = (x - r)q_1(x)$, where $q_1(x)$ is a polynomial of degree one less than that of $q(x)$ and such that $q_1(r) \neq 0$, that is, $q_1(x)$ does not contain $x - r$ as a factor. Thus, we have the identity

$$\frac{p(x)}{q(x)} = \frac{p(x)}{(x - r)q_1(x)}.$$

In this case, we shall show that $p(x)/q(x)$ has a corresponding partial fraction $A/(x - r)$, where A is a nonzero constant.

Let us consider the problem of determining A so that the difference

$$\frac{p(x)}{(x - r)q_1(x)} - \frac{A}{x - r}$$

shall combine into a fraction that can be reduced by dividing numerator and denominator by $x - r$. Combining the two fractions, we obtain

$$\frac{p(x) - Aq_1(x)}{(x - r)q_1(x)},$$

where the numerator is a new polynomial. Hence, if this numerator has the factor $x - r$, it must be zero for $x = r$, that is,

$$p(r) - Aq_1(r) = 0,$$

or
$$A = \frac{p(r)}{q_1(r)}. \tag{1}$$
Since $x - r$ is not a factor of $p(x)$, we know that $p(r)$ cannot be zero and A is a nonzero constant. Using the preceding value of A, we have
$$\frac{p(x) - Aq_1(x)}{(x - r)q_1(x)} = \frac{(x - r)p_1(x)}{(x - r)q_1(x)} = \frac{p_1(x)}{q_1(x)}.$$
Furthermore, the degree of $p(x)$ is by hypothesis less than that of $(x - r)q_1(x)$ so that $p(x) - Aq_1(x)$ is of no higher degree than $q_1(x)$. Consequently, $p_1(x)$ is of lower degree than $q_1(x)$, that is, $p_1(x)/q_1(x)$ is also a proper fraction. We thus have the result that
$$\frac{p(x)}{(x - r)q_1(x)} = \frac{A}{x - r} + \frac{p_1(x)}{q_1(x)}.$$

It should now be clear that, if $q_1(x)$ has a simple linear factor, the same procedure will serve to split off another partial fraction from the fraction $p_1(x)/q_1(x)$. Indeed, if the original denominator, $q(x)$, has only simple linear factors, we can continue the procedure until $p(x)/q(x)$ is completely decomposed into partial fractions, all of the type $A/(x - r)$.

Now, suppose that
$$\frac{p(x)}{q(x)} = \frac{p(x)}{b_0(x - r_1)(x - r_2)\cdots(x - r_n)},$$
where the r's are all distinct. We have shown that there exists a set of partial fractions such that
$$\frac{p(x)}{q(x)} = \frac{A_1}{x - r_1} + \frac{A_2}{x - r_2} + \cdots + \frac{A_n}{x - r_n}.$$
It is a fact, although we omit the proof, that there is one and only one set of values for the A's regardless of the method or the order used to find them. Consequently, each A may be obtained from the original fraction by a formula corresponding to Formula (1). If we let $q_k(x)$ denote the polynomial resulting from $q(x)$ when the factor $x - r_k$ is deleted, then
$$A_k = \frac{p(r_k)}{q_k(r_k)}. \tag{2}$$

EXAMPLE 1. Decompose into partial fractions:
$$\frac{x^2 - 3x + 4}{(x - 1)(x + 1)(x + 2)}.$$

Solution: First Method: We take $r_1 = 1$, $r_2 = -1$, and $r_3 = -2$. Then

$$q_1(x) = (x+1)(x+2),$$
$$q_2(x) = (x-1)(x+2),$$
$$q_3(x) = (x-1)(x+1),$$

and, by the use of Formula (2),

$$A_1 = \frac{p(1)}{q_1(1)} = \frac{1-3+4}{(1+1)(1+2)} = \frac{1}{3},$$

$$A_2 = \frac{p(-1)}{q_2(-1)} = \frac{1+3+4}{(-1-1)(-1+2)} = -4,$$

and

$$A_3 = \frac{p(-2)}{q_3(-2)} = \frac{4+6+4}{(-2-1)(-2+1)} = \frac{14}{3}.$$

Therefore,

$$\frac{x^2-3x+4}{(x-1)(x+1)(x+2)} = \frac{\frac{1}{3}}{x-1} - \frac{4}{x+1} + \frac{\frac{14}{3}}{x+2}. \qquad \textit{Ans.}$$

The student may combine the three partial fractions as a check on the result.

Second Method (Method of Undetermined Coefficients): It follows from our general discussion that the given fraction can be decomposed as follows:

$$\frac{x^2-3x+4}{(x-1)(x+1)(x+2)} = \frac{A_1}{x-1} + \frac{A_2}{x+1} + \frac{A_3}{x+2},$$

where A_1, A_2, and A_3 are constants to be evaluated. In order to effect this evaluation, we first multiply both members of the equation by the original denominator to get

$$x^2 - 3x + 4 = A_1(x+1)(x+2) + A_2(x-1)(x+2) + A_3(x-1)(x+1).$$

This equation is to be valid for all values of x except possibly the values 1, -1, and -2 (which make the original denominator zero). Hence, it follows from Theorem 3, Section 110, that the equation is an identity valid even for the values $x = 1, -1$, and -2.

By putting $x = 1$ in both members, we knock out in the right member all but the A_1 term, and find $2 = 6A_1$, or $A_1 = \frac{1}{3}$. Similarly, by using $x = -1$ and $x = -2$, we find $A_2 = -4$ and $A_3 = \frac{14}{3}$, as in the preceding method. If the form of the partial fraction expansion is assumed, this second method can be used to derive Formula (2). The

method of undetermined coefficients is especially important in the later sections of this chapter.

In place of taking special values of x, we may compare coefficients on both sides of the identity as in Section 110. We have

$$x^2 - 3x + 4 = (A_1 + A_2 + A_3)x^2 + (3A_1 + A_2)x + (2A_1 - 2A_2 - A_3),$$

so that

$$\begin{aligned} 1 &= A_1 + A_2 + A_3, \\ -3 &= 3A_1 + A_2, \\ 4 &= 2A_1 - 2A_2 - A_3. \end{aligned}$$

The student may show that this system has the solution $A_1 = \frac{1}{3}$, $A_2 = -4$, $A_3 = \frac{14}{3}$, previously found.

147. Repeated Linear Factors

If $q(x)$ has a factor $x - r$ occurring more than once, this factor is termed a **repeated** or **multiple** linear factor. The highest power to which such a factor occurs is called its **multiplicity.** For example, in the polynomial

$$x^4 - 2x^3 + 2x - 1 = (x - 1)^3(x + 1),$$

the factor $x - 1$ is a repeated factor of multiplicity 3.

When the denominator of our rational fraction has a repeated factor, we can proceed in a manner similar to that in the preceding section. Thus, let $x - r$ be a factor of multiplicity $k > 1$. Then

$$q(x) = (x - r)^k q_1(x), \text{ where } q_1(r) \neq 0,$$

and

$$\frac{p(x)}{q(x)} = \frac{p(x)}{(x - r)^k q_1(x)}.$$

We show here that $p(x)/q(x)$ has the partial fraction $A_1/(x - r)^k$, where A_1 is a nonzero constant.

As before, we form the difference

$$\frac{p(x)}{(x - r)^k q_1(x)} - \frac{A_1}{(x - r)^k} = \frac{p(x) - A_1 q_1(x)}{(x - r)^k q_1(x)},$$

and ask that the polynomial in the numerator have $x - r$ as a factor. (Notice that we do not require this factor to be more than a simple one, although in special cases it may turn out to be of multiplicity 2 or more.) Again, we must have

$$p(r) - A_1 q_1(r) = 0,$$

or
$$A_1 = \frac{p(r)}{q_1(r)}, \tag{1}$$
where it is to be emphasized that $q_1(x)$ is the polynomial resulting from $p(x)$ when the factor $(x - r)^k$ is deleted. Since $p(r) \neq 0$, A_1 is a nonzero constant as we predicted.

The value of A_1 just found may be used to write
$$\frac{p(x) - A_1 q_1(x)}{(x - r)^k q_1(x)} = \frac{(x - r)p_1(x)}{(x - r)^k q_1(x)} = \frac{p_1(x)}{(x - r)^{k-1} q_1(x)}.$$
Thus, we have
$$\frac{p(x)}{(x - r)^k q_1(x)} = \frac{A_1}{(x - r)^k} + \frac{p_1(x)}{(x - r)^{k-1} q_1(x)}.$$
It is left to the student to show that the second term in the right member of this identity is a proper fraction.

Now, if $p_1(x)$ does not have $x - r$ as a factor, then in the second fraction on the right the factor $x - r$ occurs $k - 1$ times in the denominator. In any case, our procedure yields a result in which the multiplicity of $x - r$ has been reduced by unity at least. Obviously, the same procedure may be applied repeatedly to split off more partial fractions until the factor $x - r$ is completely removed. The end result will be
$$\frac{p(x)}{(x - r)^k q_1(x)} = \frac{A_1}{(x - r)^k} + \frac{A_2}{(x - r)^{k-1}} + \cdots + \frac{A_k}{x - r} + \frac{p_k(x)}{q_1(x)},$$
where the last fraction is a proper one to which the same methods may be applied. We do not attempt to give formulas for $A_2, A_3, \cdots, A_k$ in this case. However, we can compute the A's one at a time in succession as is indicated by the preceding discussion. Alternatively, we can use a second method which will be explained by means of the next example. It is again of importance for us to know that the A's are unique, independent of the method used to find them. We shall not prove this fact. Notice that A_1 cannot be zero, but any of the other A's may be zero in special cases.

Example 1. Expand into partial fractions: $\dfrac{x^2 + 4}{(x - 1)^2(x - 2)^2}$.

Solution—First Method: We know that there are partial fractions $A_1/(x - 1)^2$ and $B_1/(x - 2)^2$, where, by Formula (1),
$$A_1 = \left.\frac{x^2 + 4}{(x - 2)^2}\right]_{x=1} = 5,$$

and
$$B_1 = \frac{x^2 + 4}{(x - 1)^2}\Bigg]_{x=2} = 8.$$

(The notation $f(x)]_{x=a}$ means the value of $f(x)$ for $x = a$.)

Now subtract the two partial fractions from the original fraction:

$$\frac{x^2 + 4}{(x - 1)^2(x - 2)^2} - \frac{5}{(x - 1)^2} - \frac{8}{(x - 2)^2} = \frac{-12(x^2 - 3x + 2)}{(x - 1)^2(x - 2)^2}$$
$$= \frac{-12}{(x - 1)(x - 2)}.$$

The last fraction can be split by the method of the preceding section to give

$$\frac{-12}{(x - 1)(x - 2)} = \frac{12}{x - 1} - \frac{12}{x - 2}.$$

Therefore,

$$\frac{x^2 + 4}{(x - 1)^2(x - 2)^2} = \frac{5}{(x - 1)^2} + \frac{12}{x - 1} + \frac{8}{(x - 2)^2} - \frac{12}{x - 2}. \quad \textit{Ans.}$$

Second Method: Our general discussion shows that the given fraction can be decomposed as follows:

$$\frac{x^2 + 4}{(x - 1)^2(x - 2)^2} = \frac{A_1}{(x - 1)^2} + \frac{A_2}{(x - 1)} + \frac{B_1}{(x - 2)^2} + \frac{B_2}{x - 2},$$

where the A's and B's are numerical coefficients whose values are to be properly determined. In order to effect this determination, we first multiply both members of the equation by the original denominator to get the identity:

$$x^2 + 4 = A_1(x - 2)^2 + A_2(x - 1)(x - 2)^2 + B_1(x - 1)^2$$
$$+ B_2(x - 1)^2(x - 2).$$

By putting $x = 1$ in both members, we knock out in the right member all except the A_1 term, and find $5 = A_1$. Similarly, by using the value 2 for x, we get $8 = B_1$. In order to determine A_2 and B_2, we may equate corresponding coefficients on both sides. We have

$$x^2 + 4 = (A_2 + B_2)x^3 + (A_1 - 5A_2 + B_1 - 4B_2)x^2$$
$$+(-4A_1 + 8A_2 - 2B_1 + 5B_2)x + (4A_1 - 4A_2 + B_1 - 2B_2),$$

so that comparison of coefficients gives the equations:

$$0 = A_2 + B_2,$$

$$1 = A_1 - 5A_2 + B_1 - 4B_2,$$

$$0 = -4A_1 + 8A_2 - 2B_1 + 5B_2,$$

$$4 = 4A_1 - 4A_2 + B_1 - 2B_2.$$

Since we know that $A_1 = 5$ and $B_1 = 8$, we need only two of these four equations. Taking the first two, and using the known values of A_1 and B_1, we find $A_2 = 12$ and $B_2 = -12$, as before. The student may check the four values in the remaining two equations to show their consistency.

Since it is frequently easy to find some of the coefficients by the use of special values of x, as in this example, it seems a good plan to pick off only the simpler terms of the identity for the remaining necessary equations. For instance, the student should be able by inspection to write the coefficients of x and the constant terms in this example to obtain the first and the last of the preceding equations, respectively.

EXERCISES 118

Resolve each of the following into partial fractions:

1. $\dfrac{3x - 5}{(x - 1)(x - 3)}$
2. $\dfrac{2y - 2}{(y + 5)(y + 2)}$
3. $\dfrac{v + 14}{v^2 + 14v + 48}$
4. $\dfrac{4r + 2}{r^2 - 4r - 21}$
5. $\dfrac{5x^2 - 30}{x^2 - x - 6}$
6. $\dfrac{u^3 + 2}{u^2 - 2u - 8}$
7. $\dfrac{9w^2 + 2w - 14}{w^2(w - 7)}$
8. $\dfrac{-3k^2 + 5k + 5}{k^2(k + 1)}$
9. $\dfrac{2z^2 - z + 8}{z(z - 2)^2}$
10. $\dfrac{x^2 + x + 29}{(x - 4)(x + 3)^2}$
11. $\dfrac{-5t + 24}{(2t - 3)(t + 4)}$
12. $\dfrac{13y - 24}{(3y - 5)(y - 2)}$
13. $\dfrac{-9v - 7}{(2v + 1)(3v + 2)}$
14. $\dfrac{2u + 19}{(2u - 5)(2u - 1)}$
15. $\dfrac{-21x + 11}{(x - 1)(x - 2)(x - 3)}$
16. $\dfrac{11m + 31}{(m + 1)(m + 2)(m + 5)}$
17. $\dfrac{7y^2 + 9y + 14}{(y - 1)(y + 4)(y + 1)}$
18. $\dfrac{v^2 - 8v + 9}{(v - 2)(v - 4)(v - 5)}$

19. $\dfrac{5z^2 - 21z + 13}{(z-3)^2(z+2)}$

20. $\dfrac{-s^2 + 13s - 26}{(s+1)^2(s-4)}$

21. $\dfrac{10w - 2}{(w+3)(w-5)^2}$

22. $\dfrac{k-3}{(k+7)(k-1)^2}$

23. $\dfrac{4}{x^2(x-2)^2}$

24. $\dfrac{32}{y^2(y+4)^2}$

25. $\dfrac{3v^2 + 14v - 37}{(v+1)^2(v-3)^2}$

26. $\dfrac{7x + 16}{(x+2)^2(x+3)^2}$

148. Quadratic Factors

Although the methods of the last two sections are not theoretically restricted to the case of linear factors that are real, the results will generally involve imaginary coefficients when the factors are not real, and hence will not be in suitable form for many of the applications. Since we are dealing with rational fractions having real coefficients, we know that imaginary factors of the denominator can occur in conjugate pairs only, that is, if $x - a - ib$ is a factor then so is $x - a + ib$. Suppose these factors to be simple ones. Then, according to our previous discussion, there will be two partial fractions $C_1/(x - a - ib)$ and $C_2/(x - a + ib)$, where C_1 and C_2 will usually be imaginary. If these two partial fractions are added, it can be shown that the result is of the form $(Ax + B)/(x^2 + cx + d)$, where all the coefficients are real. As a consequence we know that, if $q(x)$ has the real quadratic factor $x^2 + cx + d$, then $p(x)/q(x)$ has the partial fraction

$$\frac{Ax + B}{x^2 + cx + d}.$$

By an argument similar to that used in Section 147, we can show that, if $x^2 + cx + d$ occurs as a repeated factor of multiplicity k, then there is the following partial fraction expansion corresponding to this factor:

$$\frac{A_1x + B_1}{(x^2 + cx + d)^k} + \frac{A_2x + B_2}{(x^2 + cx + d)^{k-1}} + \cdots + \frac{A_kx + B_k}{x^2 + cx + d}.$$

This expansion is completely analogous to the one for a repeated linear factor. The method of undetermined coefficients is usually used for the determination of the A's and the B's. It is to be understood that the denominator of the original rational fraction may be any combination of real linear and quadratic factors, simple or repeated. The complete partial fraction expansion will consist of the sum of the various fractions which are characteristic of the respective factors.

EXAMPLE 1. Decompose into partial fractions: $\dfrac{4x - 16}{(x + 2)^2(x^2 + 4)}$.

Solution: We know that

$$\frac{4x - 16}{(x + 2)^2(x^2 + 4)} = \frac{A_1}{(x + 2)^2} + \frac{A_2}{x + 2} + \frac{Bx + C}{x^2 + 4}.$$

Multiplying both members by the original denominator, we get the identity

$$4x - 16 = A_1(x^2 + 4) + A_2(x + 2)(x^2 + 4) + (Bx + C)(x + 2)^2.$$

Putting $x = -2$, we find

$$-24 = 8A_1 \qquad \text{or} \qquad A_1 = -3.$$

By collecting the coefficients of the following indicated terms, we obtain three equations for the remaining constants.

$$\begin{aligned} x^3: &\quad 0 = A_2 + B, \\ x^2: &\quad 0 = A_1 + 2A_2 + 4B + C, \\ x^0: &\quad -16 = 4A_1 + 8A_2 + 4C. \end{aligned}$$

We solve these three equations and collect results:

$$A_1 = -3, \quad A_2 = -1, \quad B = 1, \quad C = 1.$$

Therefore,

$$\frac{4x - 16}{(x + 2)^2(x^2 + 4)} = -\frac{3}{(x + 2)^2} - \frac{1}{x + 2} + \frac{x + 1}{x^2 + 4}. \quad \textit{Ans.}$$

EXAMPLE 2. Expand into partial fractions: $\dfrac{x^3 + x^2 + 3x - 3}{(x^2 + 1)(x^2 + 2x + 2)}$.

Solution: We have

$$\frac{x^3 + x^2 + 3x - 3}{(x^2 + 1)(x^2 + 2x + 2)} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{x^2 + 2x + 2},$$

and, thus, can write the identity

$$x^3 + x^2 + 3x - 3 = (Ax + B)(x^2 + 2x + 2) + (Cx + D)(x^2 + 1).$$

Equating coefficients, we obtain the system of equations

$$\begin{aligned} x^3: &\quad 1 = A + C, \\ x^2: &\quad 1 = 2A + B + D, \\ x\ : &\quad 3 = 2A + 2B + C, \\ x^0: &\quad -3 = 2B + D. \end{aligned}$$

This system has the solution $A = 2$, $B = 0$, $C = -1$, $D = -3$ so that

$$\frac{x^3 + x^2 + 3x - 3}{(x^2 + 1)(x^2 + 2x + 2)} = \frac{2x}{x^2 + 1} - \frac{x + 3}{x^2 + 2x + 2}. \quad \textit{Ans.}$$

EXAMPLE 3. Expand into partial fractions: $\dfrac{8}{x(x^2 + 2)^2}$.

Solution: We write

$$\frac{8}{x(x^2 + 2)^2} = \frac{A}{x} + \frac{Bx + C}{(x^2 + 2)^2} + \frac{Dx + E}{x^2 + 2}.$$

When both sides are multiplied by $x(x^2 + 2)^2$, the resulting identity is

$$8 = A(x^2 + 2)^2 + x(Bx + C) + x(x^2 + 2)(Dx + E).$$

The value $x = 0$ gives $8 = 4A$ or $A = 2$. By equating coefficients, we get

$$\begin{aligned} x^4&: \quad 0 = A + D, \\ x^3&: \quad 0 = E, \\ x^2&: \quad 0 = 4A + B + 2D, \\ x\ &: \quad 0 = C + 2E. \end{aligned}$$

With $A = 2$, these equations have the solution

$$B = -4, \quad C = 0, \quad D = -2, \quad E = 0.$$

Accordingly,

$$\frac{8}{x(x^2 + 2)^2} = \frac{2}{x} - \frac{4x}{(x^2 + 2)^2} - \frac{2x}{x^2 + 2}. \quad \textit{Ans.}$$

EXERCISES 119

Resolve each of the following into partial fractions:

1. $\dfrac{6}{x(x^2 + 3)}$

2. $\dfrac{30v}{(v - 6)(v^2 + 4)}$

3. $\dfrac{9y^2 + 29}{(y - 2)(y^2 + 2y + 5)}$

4. $\dfrac{15s}{s^3 + 8}$

5. $\dfrac{u^3 + 5u^2 + 14}{u^3 + 7u}$

6. $\dfrac{2x^3 + x - 15}{x^3 - x^2 + 3x - 3}$

7. $\dfrac{8}{(2w - 1)(4w^2 + 1)}$

8. $\dfrac{9t^2 + 35t}{(5t - 3)(3t^2 + 7)}$

9. $\dfrac{x^2 + 1}{(2x^2 + x + 4)(2x - 1)}$

10. $\dfrac{6k^2 + k + 1}{(k + 3)(3k^2 + k + 2)}$

11. $\dfrac{r^3 + 4}{(r - 1)(r + 2)(r^2 + 3r + 1)}$

12. $\dfrac{3u^3 + u - 4}{u(u + 2)(u^2 + 2u + 5)}$

13. $\dfrac{5y^2 + 7}{y^4 + y^2 - 2}$

14. $\dfrac{10x + 6}{x^4 - 8x^2 - 9}$

15. $\dfrac{3x^3 + x^2 + 11x - 15}{(x^2 + 1)(x^2 + 9)}$

16. $\dfrac{z^3 + 5z + 12}{(z^2 + 2)(z^2 + 5)}$

17. $\dfrac{2v^3 + 4}{v^4 + 1}$

18. $\dfrac{6}{y^6 - 1}$

19. $\dfrac{2m^3 - 4m + 37}{(m - 2)^2(m^2 + 5)}$

20. $\dfrac{4x^2 - 2}{(x + 1)^2(x^2 + 1)^2}$

21. $\dfrac{2x^3 - x^2 - 21}{(x^2 + x + 7)^2}$

22. $\dfrac{5v^2 - 18v}{(v^2 - 3v + 1)^2}$

23. $\dfrac{3s^3 + 6s^2 + 13s + 22}{(s - 2)(s^2 + 4)^2}$

24. $\dfrac{4x^2}{(x + 1)(x^2 + 1)^2}$

Chapter 22

FINITE DIFFERENCES

149. Introduction

In this chapter we shall consider the topic of finite differences, which is concerned with a study of the change that takes place in a function when its independent variable is increased or decreased by a given amount. We shall discuss the application of finite differences to certain series that are generalizations of the arithmetic series, and we shall describe the application to a simple interpolation problem.

It is beyond the scope of this book to give more than a brief introduction to the simpler algebraic notions connected with finite differences. However, it is interesting to know that one of the most outstanding present-day uses of this topic is as a means of writing practical approximate equations for many vital physical problems. The employment of differences in such problems most frequently involves an enormous amount of numerical calculation so that the method is severely handicapped without highly efficient means of computation. The advent of modern electronic computing devices has made the present importance of difference methods difficult to overestimate.

150. Higher Order Arithmetic Progressions

In Section 86 we studied simple arithmetic progressions, finding them to be characterized by the fact that the difference between each term and its predecessor is a constant, the so-called common difference. We shall now consider a certain type of progression where there is no constant difference between successive terms but where the sequence of differences themselves is a simple arithmetic progression. For ex-

ample, let the progression be

$$a_0 = 1, \quad a_1 = 4, \quad a_2 = 9, \quad a_3 = 16, \quad a_4 = 25,$$

where, as a matter of convenience, the first term has been numbered $n = 0$ rather than $n = 1$. Writing the sequence of differences

$$d_0 = a_1 - a_0 = 3, \quad d_1 = a_2 - a_1 = 5, \quad d_2 = a_3 - a_2 = 7,$$

$$d_3 = a_4 - a_3 = 9,$$

we see that, although the sequence of a's is not a simple arithmetic progression, the sequence of d's is exactly such a progression; there is the common difference 2 between each pair of consecutive elements. Because the differences of the differences are constant, we call the sequence of a's a **second-order arithmetic progression.** Clearly, we may similarly define third-order, fourth-order, and generally higher-order progressions.

In order to deal more conveniently with these higher order progressions, it is best to introduce here the notation of finite differences. We denote the difference $a_{n+1} - a_n$ by the symbol Δa_n, read "delta-a_n," the symbol Δ being the standard one used to symbolize a difference. Notice that Δa_n does *not* mean Δ times a_n, but that

$$\Delta a_n = a_{n+1} - a_n.$$

Thus, for the progression of a's in the preceding illustration we have

$$\Delta a_0 = 3, \quad \Delta a_1 = 5, \quad \Delta a_2 = 7, \quad \Delta a_3 = 9.$$

To go on, we denote the second differences by the symbol Δ^2, that is,

$$\Delta^2 a_n = \Delta(\Delta a_n) = \Delta a_{n+1} - \Delta a_n.$$

Again, for the illustrative sequence of a's under consideration,

$$\Delta^2 a_0 = 2, \quad \Delta^2 a_1 = 2, \quad \Delta^2 a_2 = 2.$$

In general, the kth order differences are denoted by Δ^k so that

$$\Delta^k a_n = \Delta(\Delta^{k-1} a_n) = \Delta^{k-1} a_{n+1} - \Delta^{k-1} a_n.$$

It follows from this definition that the symbol Δ^k obeys the law of exponents in multiplication. If j and k are both positive integers, then

$$\Delta^j(\Delta^k x_n) = \Delta^k(\Delta^j x_n) = \Delta^{j+k} x_n.$$

In the table at the top of the opposite page is shown a third order progression of five terms and its corresponding differences.

n	a_n	Δa_n	$\Delta^2 a_n$	$\Delta^3 a_n$
0	2			
		3		
1	5		1	
		4		-10
2	9		-9	
		-5		-10
3	4		-19	
		-24		
4	-20			

Notice within the table the triangular arrangement wherein each difference is written on a line between the two items from which it is formed. This is a standard method for setting up such a table.

Suppose that we should like to extend the foregoing progression of a's by adjoining a few more terms. We can build up our table stepwise by working backward from the third differences. Thus, in the next table, we have repeated the last item in each column of the last table, and have then adjoined another item, -10, in the third difference column. The next item, -29, in the second difference column is obtained by *adding* the third difference -10 to the last given second difference -19. Then the -29 is added to the last given difference -24 to get the next one, -53, and, finally, the -53 is added to a_4, -20, to get $a_5 = -73$. In each case, an item from one column is added to the item to the left and just above it in the column to the left to get the next item in this left-hand column. The table may be extended as far as we like by following this same procedure repeatedly.

n	a_n	Δa_n	$\Delta^2 a_n$	$\Delta^3 a_n$
				-10
			-19	
		-24		-10
4	-20		-29	
		-53		
5	-73			

Although the schedule given in the last paragraph would not be too laborious if only one or two additional terms of the progression were

desired, it would be highly inefficient if, say, the 50th term were wanted. For this reason and also because the result is of interest in other respects, we derive a formula for the general term in terms of the first term a_0 and the differences $\Delta^j a_0$, $j = 1, 2, 3, \cdots$.

We have by definition

$$a_1 = a_0 + \Delta a_0,$$

and

$$a_2 = a_1 + \Delta a_1.$$

Upon substituting from the first into the second of these equations, we get

$$\begin{aligned} a_2 &= a_0 + \Delta a_0 + \Delta(a_0 + \Delta a_0) \\ &= a_0 + \Delta a_0 \\ &\qquad + \Delta a_0 + \Delta^2 a_0, \end{aligned}$$

or

$$a_2 = a_0 + 2\Delta a_0 + \Delta^2 a_0.$$

Next,

$$\begin{aligned} a_3 &= a_2 + \Delta a_2 \\ &= a_0 + 2\Delta a_0 + \Delta^2 a_0 + \Delta(a_0 + 2\Delta a_0 + \Delta^2 a_0) \\ &= a_0 + 2\Delta a_0 + \Delta^2 a_0 \\ &\qquad + \Delta a_0 + 2\Delta^2 a_0 + \Delta^3 a_0, \end{aligned}$$

or

$$a_3 = a_0 + 3\Delta a_0 + 3\Delta^2 a_0 + \Delta^3 a_0.$$

In these results, the coefficients in the right members are exactly the binomial coefficients corresponding to the index in the left member. Furthermore, a comparison of the formation of the coefficients in general with that of the binomial coefficients in Section 81 shows that the two are the same. Thus, it follows by mathematical induction, all details being as in the proof of the binomial formula, that the desired general formula is

$$a_n = a_0 + \binom{n}{1}\Delta a_0 + \binom{n}{2}\Delta^2 a_0 + \cdots + \binom{n}{n}\Delta^n a_0. \tag{1}$$

If we detach the Δ from the a_0, it is easy to write a simple symbolic formula for this relation:

$$a_n = (1 + \Delta)^n a_0, \tag{2}$$

where, after expanding, each power of Δ is to apply to the a_0 as in Formula (1).

Since we have made no use of the constancy of any order of differences, these formulas apply even when the progression is not an arith-

metic one. If n is a positive integer, which is inherent in the derivation that we have employed, the formula, as in the case of the binomial formula terminates with the $(n + 1)$th term, $\Delta^n a_0$, unless the kth differences are constant and $k < n$. In the latter case, the formula terminates with the kth difference term, $\binom{n}{k} \Delta^k a_0$.

With these formulas, it is easy to obtain the 50th term of the third-order progression described in the preceding table. With constant third differences, we may write

$$a_n = a_0 + \binom{n}{1} \Delta a_0 + \binom{n}{2} \Delta^2 a_0 + \binom{n}{3} \Delta^3 a_0. \tag{3}$$

We must remember that the first term corresponds to $n = 0$, so the 50th term is

$$\begin{aligned} a_{49} &= a_0 + \binom{49}{1} \Delta a_0 + \binom{49}{2} \Delta^2 a_0 + \binom{49}{3} \Delta^3 a_0 \\ &= 2 + \frac{49}{1}(3) + \frac{(49)(48)}{2!}(1) + \frac{(49)(48)(47)}{3!}(-10) \\ &= 2 + 147 + 1176 - 184{,}240 = -182{,}915. \end{aligned}$$

In fact, we can do even better than this; we can write a formula for the general term a_n in terms of n alone. Thus, by substituting for a_0 and its differences in Equation (3), we find

$$\begin{aligned} a_n &= 2 + \binom{n}{1}(3) + \binom{n}{2}(1) + \binom{n}{3}(-10) \\ &= 2 + 3n + \frac{n(n-1)}{2} - \frac{10n(n-1)(n-2)}{6} \\ &= \tfrac{1}{6}(12 - 5n + 33n^2 - 10n^3), \end{aligned}$$

a third degree polynomial in n.

Our discussion now makes evident the following important fact. *The general term* a_n *of any* k*th order arithmetic progression is a* k*th degree polynomial in* n. (That the degree is exactly k is easy to see since the kth order difference $\Delta^k a_0$ cannot be zero.)

It is not difficult to prove that the converse of the preceding result is also a valid theorem. If it is shown that the first difference of a kth degree polynomial in n is of degree $k - 1$, then the desired theorem follows by repeated application of the differencing process. Accordingly, let

$$a_n = c_0 n^k + c_1 n^{k-1} + \cdots + c_{k-1} n + c_k,$$

so that

$$a_{n+1} = c_0(n + 1)^k + c_1(n + 1)^{k-1} + \cdots + c_{k-1}(n + 1) + c_k,$$

and

$$\begin{aligned}\Delta a_n &= a_{n+1} - a_n \\ &= c_0[(n + 1)^k - n^k] + c_1[(n + 1)^{k-1} - n^{k-1}] + \cdots \\ &\qquad + c_{k-1}[(n + 1) - n].\end{aligned}$$

It is now clear that when the last expression is simplified, the result will be a polynomial of degree $k - 1$. The converse theorem follows as an immediate consequence of this fact.

Our next example illustrates the use of differences to construct a polynomial curve of degree not exceeding k that passes through $k + 1$ points with equally spaced ordinates.

EXAMPLE 1. Find the cubic curve $y = c_0x^3 + c_1x^2 + c_2x + c_3$ that passes through the four points: (1, 0), (3, -44), (5, -40), and (7, 108).

Solution; In order to fit the values of x into our difference scheme, let us put $x = b + nh$, where b is the starting value of x, and h is the common interval between successive values of x. In this example, $b = 1$, $h = 2$, and the values of x correspond to $n = 0, 1, 2, 3$, in succession. The table of differences follows:

n	$x = 2n + 1$	a_n	Δa_n	$\Delta^2 a_n$	$\Delta^3 a_n$
0	1	0			
			-44		
1	3	-44		48	
			4		96
2	5	-40		144	
			148		
3	7	108			

The general formula for a_n is then

$$\begin{aligned}a_n &= a_0 + \binom{n}{1}\Delta a_0 + \binom{n}{2}\Delta^2 a_0 + \binom{n}{3}\Delta^3 a_0 \\ &= 0 + \binom{n}{1}(-44) + \binom{n}{2}(48) + \binom{n}{3}(96) \\ &= 16n^3 - 24n^2 - 36n.\end{aligned}$$

We now write y in place of a_n and substitute $n = \frac{1}{2}(x - 1)$ to get

$$y = 2x^3 - 12x^2 + 10. \quad Ans.$$

The correctness of this answer follows from the fact that y does have the required values for $n = 0, 1, 2, 3$, that is, for $x = 1, 3, 5, 7$. Also there can be not more than one polynomial curve of degree k passing through $k + 1$ distinct points, although it may happen that a curve of lower degree passes through these points. This last situation will appear from the table of differences when the items in a column of differences of order less than k are constant.

151. The Sum of a Higher Order Progression

A simple little trick enables us to find the formula for the sum of the first n terms of an arithmetic progression of order k. We first write S_n for this sum so that

$$S_n = a_0 + a_1 + a_2 + \cdots + a_{n-1}, \quad n = 1, 2, \cdots,$$

and

$$S_{n+1} = a_0 + a_1 + a_2 + \cdots + a_{n-1} + a_n.$$

It is then immediately obvious that

$$\Delta S_n = S_{n+1} - S_n = a_n, \quad n = 1, 2, \cdots,$$

and the sequence of S's has the sequence of a's, starting with a_1, as its first differences. In order to make the sequence of differences begin with a_0, we put $S_0 = 0$, which gives

$$\Delta S_0 = S_1 - S_0 = a_0.$$

We may now write $\Delta^2 S_0 = \Delta a_0$, $\Delta^3 S_0 = \Delta^2 a_0$, and so on.

Thus, by the use of the formula for the general term of a higher order progression, we have

$$S_n = S_0 + \binom{n}{1}\Delta S_0 + \binom{n}{2}\Delta^2 S_0 + \cdots + \binom{n}{k+1}\Delta^{k+1} S_0.$$

The formula stops with the $(k + 1)$th difference term because the order of the progression of S's is one greater than that of the a's. Putting in the values $S_0 = 0$, $\Delta S_0 = a_0$, and, in general,

$$\Delta^{m+1} S_0 = \Delta^m a_0, \quad m = 1, 2, \cdots, k,$$

we get

$$S_n = \binom{n}{1} a_0 + \binom{n}{2}\Delta a_0 + \binom{n}{3}\Delta^2 a_0 + \cdots + \binom{n}{k+1}\Delta^k a_0, \quad (1)$$

as the desired formula.

Example 1. Find the formula for the sum of the cubes of the first n integers.

Solution: We are to sum the third order progression

$$1^3 + 2^3 + \cdots + n^3.$$

Accordingly, we construct the following table of differences, where $a_{n-1} = n^3$:

n	a_n	Δa_n	$\Delta^2 a_n$	$\Delta^3 a_n$
0	1			
		7		
1	8		12	
		19		6
2	27		18	
		37		
3	64			

Then, by Formula (1),

$$\begin{aligned} S_n &= \binom{n}{1} a_0 + \binom{n}{2} \Delta a_0 + \binom{n}{3} \Delta^2 a_0 + \binom{n}{4} \Delta^3 a_0 \\ &= \frac{n}{1}(1) + \frac{n(n-1)}{2}(7) + \frac{n(n-1)(n-2)}{6}(12) \\ &\qquad + \frac{n(n-1)(n-2)(n-3)}{24}(6). \end{aligned}$$

After simplification, the final result is

$$S_n = 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}. \quad Ans.$$

152. Interpolation

As we have seen in connection with logarithms, linear interpolation (interpolation by proportional parts) is sufficiently accurate for a table or a portion of one in which the first differences are very nearly constant. It frequently happens that, although the first differences are not nearly constant, some higher order differences may be. In such a case, the interpolation may be performed with a higher order difference formula.

We know that, if the kth order differences are actually constant, we

can construct a polynomial curve of degree k passing through $k + 1$ points determined by $k + 1$ consecutive tabular entries, and this curve will fit the tabulated function exactly. If the differences are only approximately constant, then the curve is only an approximate one. However, this curve usually fits the function closely enough so that intermediate values may be obtained from the difference formula with sufficient accuracy for practical computation. The method will be explained by means of the next example.

EXAMPLE 1. Using Table I of the Appendix, find $\sqrt{1.525}$.

Solution: We first construct the difference table using entries from Table I starting with $N = 1.5$.

n	N	$a_n = \sqrt{N}$	Δa_n	$\Delta^2 a_n$	$\Delta^3 a_n$	$\Delta^4 a_n$
0	1.5	1.22474				
			4017			
1	1.6	1.26491		−124		
			3893		11	
2	1.7	1.30384		−113		−2
			3780		9	
3	1.8	1.34164		−104		0
			3676		9	
4	1.9	1.37840		−95		−2
			3581		7	
5	2.0	1.41421		−88		
			3493			
6	2.1	1.44914				

Decimal points have been omitted in the columns of differences. We see that the fourth differences, −0.00002, 0.00000, −0.00002, vary so little that they may be assumed nearly constant.

Writing $N = b + nh$, we have $b = 1.5$, the starting value, and $h = 0.1$, the tabular interval. Consequently, in order to find $\sqrt{1.525}$, we must choose $n = 0.25$. Then, using our formula for the general term of a higher order progression, we have

$$\sqrt{1.525} = a_0 + \binom{0.25}{1} \Delta a_0 + \binom{0.25}{2} \Delta^2 a_0 + \binom{0.25}{3} \Delta^3 a_0 + \binom{0.25}{4} \Delta^4 a_0.$$

Hence upon utilizing the differences tabulated in the preceding schedule, we get

$$\sqrt{1.525} = 1.22474 + (0.25)(0.04017) + \frac{(0.25)(-0.75)}{2}(-0.00124)$$
$$+ \frac{(0.25)(-0.75)(-1.75)}{6}(0.00011)$$
$$+ \frac{(0.25)(-0.75)(-1.75)(-2.75)}{24}(-0.00002)$$
$$= 1.22474 + 0.0100425 + 0.0001162 + 0.0000060 + 0.0000008$$
$$= 1.23491. \quad \textit{Ans.}$$

This result can be shown to be correct to five decimal places. If first differences alone were used, the result would be correct to only three decimal places.

EXERCISES 120

For each of the given progressions, form the table of differences and extend it far enough to find the next two terms:

1. 24, 22, 20, 24, 40
2. -10, -5, 0, 17, 58
3. -52, -1, 14, 11, 18
4. 18, 17, 10, -9, -46

In Numbers 5 to 8, find the equation of the cubic $y = c_0x^3 + c_1x^2 + c_2x + c_3$ which passes through the given points.

5. $(0, -10)$, $(1, -7)$, $(2, 2)$, $(3, 23)$
6. $(2, 8)$, $(3, 24)$, $(4, 58)$, $(5,116)$
7. $(-3, -56)$, $(-1, 10)$, $(1, 4)$, $(3, 70)$
8. $(-2, -27)$, $(0, -15)$, $(2, 5)$, $(4, 129)$

Find the equation of the quartic $y = c_0x^4 + c_1x^3 + c_2x^2 + c_3x + c_4$ which passes through the given points.

9. $(-1, 13)$, $(0, 7)$, $(1, 3)$, $(2, 13)$, $(3, 73)$
10. $(-2, 4)$, $(-1, -9)$, $(0, -8)$, $(1, -5)$, $(2, 12)$
11. $(-2, 21)$, $(0, 13)$, $(2, 13)$, $(4, 45)$, $(6, 229)$
12. $(-6, 565)$, $(-4, 85)$, $(-2, -11)$, $(0, -11)$, $(2, -11)$

Find the nth term, a_{n-1}, and the sum of n terms for each of the following progressions:

13. 18, 48, 90, 144, 210
14. 2, 6, 12, 20, 30
15. 4, 10, 18, 28, 40
16. 5, 12, 21, 32, 45
17. Show that the sum of the squares of the first n positive integers is

$$(n/6)(n + 1)(2n + 1).$$

18. Show that the sum of the squares of the first n positive odd integers is

$$(n/3)(2n + 1)(2n - 1).$$

19. Show that the sum of the cubes of the first n positive odd integers is

$$n^2(2n^2 - 1).$$

In Numbers 20 to 22 find a formula for the sum:

20. $2 \cdot 3 + 3 \cdot 6 + 4 \cdot 11 + \cdots + (n + 1)(n^2 + 2)$.

21. $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + \cdots + n(n + 1)(n + 2)$.

22. $1 \cdot 3 \cdot 5 + 3 \cdot 5 \cdot 7 + 5 \cdot 7 \cdot 9 + \cdots + (2n - 1)(2n + 1)(2n + 3)$.

With the aid of finite differences and Table I of the Appendix calculate the following:

23. $\sqrt{2.475}$ **24.** $\sqrt{4.225}$ **25.** $\sqrt{1.498}$

26. $\sqrt{53.72}$ **27.** $\sqrt[3]{3.855}$ **28.** $\sqrt[3]{117.6}$

Assume that you are able to find the logarithm of a number with two digits only; for example, you may look up log 2.1, log 2.2, log 2.3, and so forth. Then with the aid of finite differences and Table II of the Appendix obtain values of the following logarithms. Check the results by reading the correct values from the table.

29. log 2.125 **30.** log 3.475 **31.** log 4.538

32. log 3.857 **33.** log 5.252 **34.** log 2.316

TABLE I
Powers, Roots, Reciprocals

N	1/N	N^2	$\sqrt{N}$	$\sqrt{10N}$	N^3	$\sqrt[3]{N}$	$\sqrt[3]{10N}$	$\sqrt[3]{100N}$
1.0	1.00000	1.00	1.00000	3.16228	1.000	1.00000	2.15443	4.64159
1.1	0.90909	1.21	1.04881	3.31662	1.331	1.03228	2.22398	4.79142
1.2	0.83333	1.44	1.09545	3.46410	1.728	1.06266	2.28943	4.93242
1.3	0.76923	1.69	1.14018	3.60555	2.197	1.09139	2.35133	5.06580
1.4	0.71429	1.96	1.18322	3.74166	2.744	1.11869	2.41014	5.19249
1.5	0.66667	2.25	1.22474	3.87298	3.375	1.14471	2.46621	5.31329
1.6	0.62500	2.56	1.26491	4.00000	4.096	1.16961	2.51984	5.42884
1.7	0.58824	2.89	1.30384	4.12311	4.913	1.19348	2.57128	5.53966
1.8	0.55556	3.24	1.34164	4.24264	5.832	1.21644	2.62074	5.64622
1.9	0.52632	3.61	1.37840	4.35890	6.859	1.23856	2.66840	5.74890
2.0	0.50000	4.00	1.41421	4.47214	8.000	1.25992	2.71442	5.84804
2.1	0.47619	4.41	1.44914	4.58258	9.261	1.28058	2.75892	5.94392
2.2	0.45455	4.84	1.48324	4.69042	10.648	1.30059	2.80204	6.03681
2.3	0.43478	5.29	1.51658	4.79583	12.167	1.32001	2.84387	6.12693
2.4	0.41667	5.76	1.54919	4.89898	13.824	1.33887	2.88450	6.21447
2.5	0.40000	6.25	1.58114	5.00000	15.625	1.35721	2.92402	6.29961
2.6	0.38462	6.76	1.61245	5.09902	17.576	1.37507	2.96250	6.38250
2.7	0.37037	7.29	1.64317	5.19615	19.683	1.39248	3.00000	6.46330
2.8	0.35714	7.84	1.67332	5.29150	21.952	1.40946	3.03659	6.54213
2.9	0.34483	8.41	1.70294	5.38516	24.389	1.42604	3.07232	6.61911
3.0	0.33333	9.00	1.73205	5.47723	27.000	1.44225	3.10723	6.69433
3.1	0.32258	9.61	1.76068	5.56776	29.791	1.45810	3.14138	6.76790
3.2	0.31250	10.24	1.78885	5.65685	32.768	1.47361	3.17480	6.83990
3.3	0.30303	10.89	1.81659	5.74456	35.937	1.48881	3.20753	6.91042
3.4	0.29412	11.56	1.84391	5.83095	39.304	1.50369	3.23961	6.97953
3.5	0.28571	12.25	1.87083	5.91608	42.875	1.51829	3.27107	7.04730
3.6	0.27778	12.96	1.89737	6.00000	46.656	1.53262	3.30193	7.11379
3.7	0.27027	13.69	1.92354	6.08276	50.653	1.54668	3.33222	7.17905
3.8	0.26316	14.44	1.94936	6.16441	54.872	1.56049	3.36198	7.24316
3.9	0.25641	15.21	1.97484	6.24500	59.319	1.57406	3.39121	7.30614
4.0	0.25000	16.00	2.00000	6.32456	64.000	1.58740	3.41995	7.36806
4.1	0.24390	16.81	2.02485	6.40312	68.921	1.60052	3.44822	7.42896
4.2	0.23810	17.64	2.04939	6.48074	74.088	1.61343	3.47603	7.48887
4.3	0.23256	18.49	2.07364	6.55744	79.507	1.62613	3.50340	7.54784
4.4	0.22727	19.36	2.09762	6.63325	85.184	1.63864	3.53035	7.60590
4.5	0.22222	20.25	2.12132	6.70820	91.125	1.65096	3.55689	7.66309
4.6	0.21739	21.16	2.14476	6.78233	97.336	1.66310	3.58305	7.71944
4.7	0.21277	22.09	2.16795	6.85565	103.823	1.67507	3.60883	7.77498
4.8	0.20833	23.04	2.19089	6.92820	110.592	1.68687	3.63424	7.82974
4.9	0.20408	24.01	2.21359	7.00000	117.649	1.69850	3.65931	7.88374
5.0	0.20000	25.00	2.23607	7.07107	125.000	1.70998	3.68403	7.93701
5.1	0.19608	26.01	2.25832	7.14143	132.651	1.72130	3.70843	7.98957
5.2	0.19231	27.04	2.28035	7.21110	140.608	1.73248	3.73251	8.04145
5.3	0.18868	28.09	2.30217	7.28011	148.877	1.74351	3.75629	8.09267
5.4	0.18519	29.16	2.32379	7.34847	157.464	1.75441	3.77976	8.14325
5.5	0.18182	30.25	2.34521	7.41620	166.375	1.76517	3.80295	8.19321
N	1/N	N^2	$\sqrt{N}$	$\sqrt{10N}$	N^3	$\sqrt[3]{N}$	$\sqrt[3]{10N}$	$\sqrt[3]{100N}$

N	1/N	N^2	$\sqrt{N}$	$\sqrt{10N}$	N^3	$\sqrt[3]{N}$	$\sqrt[3]{10N}$	$\sqrt[3]{100N}$
5.5	0.18182	30.25	2.34521	7.41620	166.375	1.76517	3.80295	8.19321
5.6	0.17857	31.36	2.36643	7.48331	175.616	1.77581	3.82586	8.24257
5.7	0.17544	32.49	2.38747	7.54983	185.193	1.78632	3.84850	8.29134
5.8	0.17241	33.64	2.40832	7.61577	195.112	1.79670	3.87088	8.33955
5.9	0.16949	34.81	2.42899	7.68115	205.379	1.80697	3.89300	8.38721
6.0	0.16667	36.00	2.44949	7.74597	216.000	1.81712	3.91487	8.43433
6.1	0.16393	37.21	2.46982	7.81025	226.981	1.82716	3.93650	8.48093
6.2	0.16129	38.44	2.48998	7.87401	238.328	1.83709	3.95789	8.52702
6.3	0.15873	39.69	2.50998	7.93725	250.047	1.84691	3.97906	8.57262
6.4	0.15625	40.96	2.52982	8.00000	262.144	1.85664	4.00000	8.61774
6.5	0.15385	42.25	2.54951	8.06226	274.625	1.86626	4.02073	8.66239
6.6	0.15152	43.56	2.56905	8.12404	287.496	1.87578	4.04124	8.70659
6.7	0.14925	44.89	2.58844	8.18535	300.763	1.88520	4.06155	8.75034
6.8	0.14706	46.24	2.60768	8.24621	314.432	1.89454	4.08166	8.79366
6.9	0.14493	47.61	2.62679	8.30662	328.509	1.90378	4.10157	8.83656
7.0	0.14286	49.00	2.64575	8.36660	343.000	1.91293	4.12129	8.87904
7.1	0.14085	50.41	2.66458	8.42615	357.911	1.92200	4.14082	8.92112
7.2	0.13889	51.84	2.68328	8.48528	373.248	1.93098	4.16017	8.96281
7.3	0.13699	53.29	2.70185	8.54400	389.017	1.93988	4.17934	9.00411
7.4	0.13514	54.76	2.72029	8.60233	405.224	1.94870	4.19834	9.04504
7.5	0.13333	56.25	2.73861	8.66025	421.875	1.95743	4.21716	9.08560
7.6	0.13158	57.76	2.75681	8.71780	438.976	1.96610	4.23582	9.12581
7.7	0.12987	59.29	2.77489	8.77496	456.533	1.97468	4.25432	9.16566
7.8	0.12821	60.84	2.79285	8.83176	474.552	1.98319	4.27266	9.20516
7.9	0.12658	62.41	2.81069	8.88819	493.039	1.99163	4.29084	9.24434
8.0	0.12500	64.00	2.82843	8.94427	512.000	2.00000	4.30887	9.28318
8.1	0.12346	65.61	2.84605	9.00000	531.441	2.00830	4.32675	9.32170
8.2	0.12195	67.24	2.86356	9.05539	551.368	2.01653	4.34448	9.35990
8.3	0.12048	68.89	2.88097	9.11043	571.787	2.02469	4.36207	9.39780
8.4	0.11905	70.56	2.89828	9.16515	592.704	2.03279	4.37952	9.43539
8.5	0.11765	72.25	2.91548	9.21954	614.125	2.04083	4.39683	9.47268
8.6	0.11628	73.96	2.93258	9.27362	636.056	2.04880	4.41400	9.50969
8.7	0.11494	75.69	2.94958	9.32738	658.503	2.05671	4.43105	9.54640
8.8	0.11364	77.44	2.96648	9.38083	681.472	2.06456	4.44796	9.58284
8.9	0.11236	79.21	2.98329	9.43398	704.969	2.07235	4.46475	9.61900
9.0	0.11111	81.00	3.00000	9.48683	729.000	2.08008	4.48140	9.65489
9.1	0.10989	82.81	3.01662	9.53939	753.571	2.08776	4.49794	9.69052
9.2	0.10870	84.64	3.03315	9.59166	778.688	2.09538	4.51436	9.72589
9.3	0.10753	86.49	3.04959	9.64365	804.357	2.10294	4.53065	9.76100
9.4	0.10638	88.36	3.06594	9.69536	830.584	2.11045	4.54684	9.79586
9.5	0.10526	90.25	3.08221	9.74679	857.375	2.11791	4.56290	9.83048
9.6	0.10417	92.16	3.09839	9.79796	884.736	2.12532	4.57886	9.86485
9.7	0.10309	94.09	3.11448	9.84886	912.673	2.13267	4.59470	9.89898
9.8	0.10204	96.04	3.13050	9.89949	941.192	2.13997	4.61044	9.93288
9.9	0.10101	98.01	3.14643	9.94987	970.299	2.14723	4.62607	9.96655
10.0	0.10000	100.00	3.16228	10.00000	1000.000	2.15443	4.64159	10.00000
N	1/N	N^2	$\sqrt{N}$	$\sqrt{10N}$	N^3	$\sqrt[3]{N}$	$\sqrt[3]{10N}$	$\sqrt[3]{100N}$

TABLE I — Powers, Roots, Reciprocals

N	0	1	2	3	4	5	6	7	8	9
100	00 000	043	087	130	173	217	260	303	346	389
101	432	475	518	561	604	647	689	732	775	817
102	860	903	945	988	*030	*072	*115	*157	*199	*242
103	01 284	326	368	410	452	494	536	578	620	662
104	703	745	787	828	870	912	953	995	*036	*078
105	02 119	160	202	243	284	325	366	407	449	490
106	531	572	612	653	694	735	776	816	857	898
107	938	979	*019	*060	*100	*141	*181	*222	*262	*302
108	03 342	383	423	463	503	543	583	623	663	703
109	743	782	822	862	902	941	981	*021	*060	*100
110	04 139	179	218	258	297	336	376	415	454	493
111	532	571	610	650	689	727	766	805	844	883
112	922	961	999	*038	*077	*115	*154	*192	*231	*269
113	05 308	346	385	423	461	500	538	576	614	652
114	690	729	767	805	843	881	918	956	994	*032
115	06 070	108	145	183	221	258	296	333	371	408
116	446	483	521	558	595	633	670	707	744	781
117	819	856	893	930	967	*004	*041	*078	*115	*151
118	07 188	225	262	298	335	372	408	445	482	518
119	555	591	628	664	700	737	773	809	846	882
120	918	954	990	*027	*063	*099	*135	*171	*207	*243
121	08 279	314	350	386	422	458	493	529	565	600
122	636	672	707	743	778	814	849	884	920	955
123	991	*026	*061	*096	*132	*167	*202	*237	*272	*307
124	09 342	377	412	447	482	517	552	587	621	656
125	691	726	760	795	830	864	899	934	968	*003
126	10 037	072	106	140	175	209	243	278	312	346
127	380	415	449	483	517	551	585	619	653	687
128	721	755	789	823	857	890	924	958	992	*025
129	11 059	093	126	160	193	227	261	294	327	361
130	394	428	461	494	528	561	594	628	661	694
131	727	760	793	826	860	893	926	959	992	*024
132	12 057	090	123	156	189	222	254	287	320	352
133	385	418	450	483	516	548	581	613	646	678
134	710	743	775	808	840	872	905	937	969	*001
135	13 033	066	098	130	162	194	226	258	290	322
136	354	386	418	450	481	513	545	577	609	640
137	672	704	735	767	799	830	862	893	925	956
138	988	*019	*051	*082	*114	*145	*176	*208	*239	*270
139	14 301	333	364	395	426	457	489	520	551	582
140	613	644	675	706	737	768	799	829	860	891
141	922	953	983	*014	*045	*076	*106	*137	*168	*198
142	15 229	259	290	320	351	381	412	442	473	503
143	534	564	594	625	655	685	715	746	776	806
144	836	866	897	927	957	987	*017	*047	*077	*107
145	16 137	167	197	227	256	286	316	346	376	406
146	435	465	495	524	554	584	613	643	673	702
147	732	761	791	820	850	879	909	938	967	997
148	17 026	056	085	114	143	173	202	231	260	289
149	319	348	377	406	435	464	493	522	551	580
150	609	638	667	696	725	754	782	811	840	869
N	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	44	43	42
1	4.4	4.3	4.2
2	8.8	8.6	8.4
3	13.2	12.9	12.6
4	17.6	17.2	16.8
5	22.0	21.5	21.0
6	26.4	25.8	25.2
7	30.8	30.1	29.4
8	35.2	34.4	33.6
9	39.6	38.7	37.8

	41	40	39
1	4.1	4.0	3.9
2	8.2	8.0	7.8
3	12.3	12.0	11.7
4	16.4	16.0	15.6
5	20.5	20.0	19.5
6	24.6	24.0	23.4
7	28.7	28.0	27.3
8	32.8	32.0	31.2
9	36.9	36.0	35.1

	38	37	36
1	3.8	3.7	3.6
2	7.6	7.4	7.2
3	11.4	11.1	10.8
4	15.2	14.8	14.4
5	19.0	18.5	18.0
6	22.8	22.2	21.6
7	26.6	25.9	25.2
8	30.4	29.6	28.8
9	34.2	33.3	32.4

	35	34	33
1	3.5	3.4	3.3
2	7.0	6.8	6.6
3	10.5	10.2	9.9
4	14.0	13.6	13.2
5	17.5	17.0	16.5
6	21.0	20.4	19.8
7	24.5	23.8	23.1
8	28.0	27.2	26.4
9	31.5	30.6	29.7

	32	31	30
1	3.2	3.1	3.0
2	6.4	6.2	6.0
3	9.6	9.3	9.0
4	12.8	12.4	12.0
5	16.0	15.5	15.0
6	19.2	18.6	18.0
7	22.4	21.7	21.0
8	25.6	24.8	24.0
9	28.8	27.9	27.0

Prop. Parts

N	0	1	2	3	4	5	6	7	8	9
150	17 609	638	667	696	725	754	782	811	840	869
151	898	926	955	984	*013	*041	*070	*099	*127	*156
152	18 184	213	241	270	298	327	355	384	412	441
153	469	498	526	554	583	611	639	667	696	724
154	752	780	808	837	865	893	921	949	977	*005
155	19 033	061	089	117	145	173	201	229	257	285
156	312	340	368	396	424	451	479	507	535	562
157	590	618	645	673	700	728	756	783	811	838
158	866	893	921	948	976	*003	*030	*058	*085	*112
159	20 140	167	194	222	249	276	303	330	358	385
160	412	439	466	493	520	548	575	602	629	656
161	683	710	737	763	790	817	844	871	898	925
162	952	978	*005	*032	*059	*085	*112	*139	*165	*192
163	21 219	245	272	299	325	352	378	405	431	458
164	484	511	537	564	590	617	643	669	696	722
165	748	775	801	827	854	880	906	932	958	985
166	22 011	037	063	089	115	141	167	194	220	246
167	272	298	324	350	376	401	427	453	479	505
168	531	557	583	608	634	660	686	712	737	763
169	789	814	840	866	891	917	943	968	994	*019
170	23 045	070	096	121	147	172	198	223	249	274
171	300	325	350	376	401	426	452	477	502	528
172	553	578	603	629	654	679	704	729	754	779
173	805	830	855	880	905	930	955	980	*005	*030
174	24 055	080	105	130	155	180	204	229	254	279
175	304	329	353	378	403	428	452	477	502	527
176	551	576	601	625	650	674	699	724	748	773
177	797	822	846	871	895	920	944	969	993	*018
178	25 042	066	091	115	139	164	188	212	237	261
179	285	310	334	358	382	406	431	455	479	503
180	527	551	575	600	624	648	672	696	720	744
181	768	792	816	840	864	888	912	935	959	983
182	26 007	031	055	079	102	126	150	174	198	221
183	245	269	293	316	340	364	387	411	435	458
184	482	505	529	553	576	600	623	647	670	694
185	717	741	764	788	811	834	858	881	905	928
186	951	975	998	*021	*045	*068	*091	*114	*138	*161
187	27 184	207	231	254	277	300	323	346	370	393
188	416	439	462	485	508	531	554	577	600	623
189	646	669	692	715	738	761	784	807	830	852
190	875	898	921	944	967	989	*012	*035	*058	*081
191	28 103	126	149	171	194	217	240	262	285	307
192	330	353	375	398	421	443	466	488	511	533
193	556	578	601	623	646	668	691	713	735	758
194	780	803	825	847	870	892	914	937	959	981
195	29 003	026	048	070	092	115	137	159	181	203
196	226	248	270	292	314	336	358	380	403	425
197	447	469	491	513	535	557	579	601	623	645
198	667	688	710	732	754	776	798	820	842	863
199	885	907	929	951	973	994	*016	*038	*060	*081
200	30 103	125	146	168	190	211	233	255	276	298
N	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	29	28
1	2.9	2.8
2	5.8	5.6
3	8.7	8.4
4	11.6	11.2
5	14.5	14.0
6	17.4	16.8
7	20.3	19.6
8	23.2	22.4
9	26.1	25.2

	27	26
1	2.7	2.6
2	5.4	5.2
3	8.1	7.8
4	10.8	10.4
5	13.5	13.0
6	16.2	15.6
7	18.9	18.2
8	21.6	20.8
9	24.3	23.4

	25
1	2.5
2	5.0
3	7.5
4	10.0
5	12.5
6	15.0
7	17.5
8	20.0
9	22.5

	24	23
1	2.4	2.3
2	4.8	4.6
3	7.2	6.9
4	9.6	9.2
5	12.0	11.5
6	14.4	13.8
7	16.8	16.1
8	19.2	18.4
9	21.6	20.7

	22	21
1	2.2	2.1
2	4.4	4.2
3	6.6	6.3
4	8.8	8.4
5	11.0	10.5
6	13.2	12.6
7	15.4	14.7
8	17.6	16.8
9	19.8	18.9

Prop. Parts

N	0	1	2	3	4	5	6	7	8	9
200	30 103	125	146	168	190	211	233	255	276	298
201	320	341	363	384	406	428	449	471	492	514
202	535	557	578	600	621	643	664	685	707	728
203	750	771	792	814	835	856	878	899	920	942
204	963	984	*006	*027	*048	*069	*091	*112	*133	*154
205	31 175	197	218	239	260	281	302	323	345	366
206	387	408	429	450	471	492	513	534	555	576
207	597	618	639	660	681	702	723	744	765	785
208	806	827	848	869	890	911	931	952	973	994
209	32 015	035	056	077	098	118	139	160	181	201
210	222	243	263	284	305	325	346	366	387	408
211	428	449	469	490	510	531	552	572	593	613
212	634	654	675	695	715	736	756	777	797	818
213	838	858	879	899	919	940	960	980	*001	*021
214	33 041	062	082	102	122	143	163	183	203	224
215	244	264	284	304	325	345	365	385	405	425
216	445	465	486	506	526	546	566	586	606	626
217	646	666	686	706	726	746	766	786	806	826
218	846	866	885	905	925	945	965	985	*005	*025
219	34 044	064	084	104	124	143	163	183	203	223
220	242	262	282	301	321	341	361	380	400	420
221	439	459	479	498	518	537	557	577	596	616
222	635	655	674	694	713	733	753	772	792	811
223	830	850	869	889	908	928	947	967	986	*005
224	35 025	044	064	083	102	122	141	160	180	199
225	218	238	257	276	295	315	334	353	372	392
226	411	430	449	468	488	507	526	545	564	583
227	603	622	641	660	679	698	717	736	755	774
228	793	813	832	851	870	889	908	927	946	965
229	984	*003	*021	*040	*059	*078	*097	*116	*135	*154
230	36 173	192	211	229	248	267	286	305	324	342
231	361	380	399	418	436	455	474	493	511	530
232	549	568	586	605	624	642	661	680	698	717
233	736	754	773	791	810	829	847	866	884	903
234	922	940	959	977	996	*014	*033	*051	*070	*088
235	37 107	125	144	162	181	199	218	236	254	273
236	291	310	328	346	365	383	401	420	438	457
237	475	493	511	530	548	566	585	603	621	639
238	658	676	694	712	731	749	767	785	803	822
239	840	858	876	894	912	931	949	967	985	*003
240	38 021	039	057	075	093	112	130	148	166	184
241	202	220	238	256	274	292	310	328	346	364
242	382	399	417	435	453	471	489	507	525	543
243	561	578	596	614	632	650	668	686	703	721
244	739	757	775	792	810	828	846	863	881	899
245	917	934	952	970	987	*005	*023	*041	*058	*076
246	39 094	111	129	146	164	182	199	217	235	252
247	270	287	305	322	340	358	375	393	410	428
248	445	463	480	498	515	533	550	568	585	602
249	620	637	655	672	690	707	724	742	759	777
250	794	811	829	846	863	881	898	915	933	950
N	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	22	21
1	2.2	2.1
2	4.4	4.2
3	6.6	6.3
4	8.8	8.4
5	11.0	10.5
6	13.2	12.6
7	15.4	14.7
8	17.6	16.8
9	19.8	18.9

	20
1	2.0
2	4.0
3	6.0
4	8.0
5	10.0
6	12.0
7	14.0
8	16.0
9	18.0

	19
1	1.9
2	3.8
3	5.7
4	7.6
5	9.5
6	11.4
7	13.3
8	15.2
9	17.1

	18
1	1.8
2	3.6
3	5.4
4	7.2
5	9.0
6	10.8
7	12.6
8	14.4
9	16.2

	17
1	1.7
2	3.4
3	5.1
4	6.8
5	8.5
6	10.2
7	11.9
8	13.6
9	15.3

Prop. Parts

N	0	1	2	3	4	5	6	7	8	9
250	39 794	811	829	846	863	881	898	915	933	950
251	967	985	*002	*019	*037	*054	*071	*088	*106	*123
252	40 140	157	175	192	209	226	243	261	278	295
253	312	329	346	364	381	398	415	432	449	466
254	483	500	518	535	552	569	586	603	620	637
255	654	671	688	705	722	739	756	773	790	807
256	824	841	858	875	892	909	926	943	960	976
257	993	*010	*027	*044	*061	*078	*095	*111	*128	*145
258	41 162	179	196	212	229	246	263	280	296	313
259	330	347	363	380	397	414	430	447	464	481
260	497	514	531	547	564	581	597	614	631	647
261	664	681	697	714	731	747	764	780	797	814
262	830	847	863	880	896	913	929	946	963	979
263	996	*012	*029	*045	*062	*078	*095	*111	*127	*144
264	42 160	177	193	210	226	243	259	275	292	308
265	325	341	357	374	390	406	423	439	455	472
266	488	504	521	537	553	570	586	602	619	635
267	651	667	684	700	716	732	749	765	781	797
268	813	830	846	862	878	894	911	927	943	959
269	975	991	*008	*024	*040	*056	*072	*088	*104	*120
270	43 136	152	169	185	201	217	233	249	265	281
271	297	313	329	345	361	377	393	409	425	441
272	457	473	489	505	521	537	553	569	584	600
273	616	632	648	664	680	696	712	727	743	759
274	775	791	807	823	838	854	870	886	902	917
275	933	949	965	981	996	*012	*028	*044	*059	*075
276	44 091	107	122	138	154	170	185	201	217	232
277	248	264	279	295	311	326	342	358	373	389
278	404	420	436	451	467	483	498	514	529	545
279	560	576	592	607	623	638	654	669	685	700
280	716	731	747	762	778	793	809	824	840	855
281	871	886	902	917	932	948	963	979	994	*010
282	45 025	040	056	071	086	102	117	133	148	163
283	179	194	209	225	240	255	271	286	301	317
284	332	347	362	378	393	408	423	439	454	469
285	484	500	515	530	545	561	576	591	606	621
286	637	652	667	682	697	712	728	743	758	773
287	788	803	818	834	849	864	879	894	909	924
288	939	954	969	984	*000	*015	*030	*045	*060	*075
289	46 090	105	120	135	150	165	180	195	210	225
290	240	255	270	285	300	315	330	345	359	374
291	389	404	419	434	449	464	479	494	509	523
292	538	553	568	583	598	613	627	642	657	672
293	687	702	716	731	746	761	776	790	805	820
294	835	850	864	879	894	909	923	938	953	967
295	982	997	*012	*026	*041	*056	*070	*085	*100	*114
296	47 129	144	159	173	188	202	217	232	246	261
297	276	290	305	319	334	349	363	378	392	407
298	422	436	451	465	480	494	509	524	538	553
299	567	582	596	611	625	640	654	669	683	698
300	712	727	741	756	770	784	799	813	828	842
N	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	18
1	1.8
2	3.6
3	5.4
4	7.2
5	9.0
6	10.8
7	12.6
8	14.4
9	16.2

	17
1	1.7
2	3.4
3	5.1
4	6.8
5	8.5
6	10.2
7	11.9
8	13.6
9	15.3

	16
1	1.6
2	3.2
3	4.8
4	6.4
5	8.0
6	9.6
7	11.2
8	12.8
9	14.4

	15
1	1.5
2	3.0
3	4.5
4	6.0
5	7.5
6	9.0
7	10.5
8	12.0
9	13.5

	14
1	1.4
2	2.8
3	4.2
4	5.6
5	7.0
6	8.4
7	9.8
8	11.2
9	12.6

Prop. Parts

N	0	1	2	3	4	5	6	7	8	9
300	47 712	727	741	756	770	784	799	813	828	842
301	857	871	885	900	914	929	943	958	972	986
302	48 001	015	029	044	058	073	087	101	116	130
303	144	159	173	187	202	216	230	244	259	273
304	287	302	316	330	344	359	373	387	401	416
305	430	444	458	473	487	501	515	530	544	558
306	572	586	601	615	629	643	657	671	686	700
307	714	728	742	756	770	785	799	813	827	841
308	855	869	883	897	911	926	940	954	968	982
309	996	*010	*024	*038	*052	*066	*080	*094	*108	*122
310	49 136	150	164	178	192	206	220	234	248	262
311	276	290	304	318	332	346	360	374	388	402
312	415	429	443	457	471	485	499	513	527	541
313	554	568	582	596	610	624	638	651	665	679
314	693	707	721	734	748	762	776	790	803	817
315	831	845	859	872	886	900	914	927	941	955
316	969	982	996	*010	*024	*037	*051	*065	*079	*092
317	50 106	120	133	147	161	174	188	202	215	229
318	243	256	270	284	297	311	325	338	352	365
319	379	393	406	420	433	447	461	474	488	501
320	515	529	542	556	569	583	596	610	623	637
321	651	664	678	691	705	718	732	745	759	772
322	786	799	813	826	840	853	866	880	893	907
323	920	934	947	961	974	987	*001	*014	*028	*041
324	51 055	068	081	095	108	121	135	148	162	175
325	188	202	215	228	242	255	268	282	295	308
326	322	335	348	362	375	388	402	415	428	441
327	455	468	481	495	508	521	534	548	561	574
328	587	601	614	627	640	654	667	680	693	706
329	720	733	746	759	772	786	799	812	825	838
330	851	865	878	891	904	917	930	943	957	970
331	983	996	*009	*022	*035	*048	*061	*075	*088	*101
332	52 114	127	140	153	166	179	192	205	218	231
333	244	257	270	284	297	310	323	336	349	362
334	375	388	401	414	427	440	453	466	479	492
335	504	517	530	543	556	569	582	595	608	621
336	634	647	660	673	686	699	711	724	737	750
337	763	776	789	802	815	827	840	853	866	879
338	892	905	917	930	943	956	969	982	994	*007
339	53 020	033	046	058	071	084	097	110	122	135
340	148	161	173	186	199	212	224	237	250	263
341	275	288	301	314	326	339	352	364	377	390
342	403	415	428	441	453	466	479	491	504	517
343	529	542	555	567	580	593	605	618	631	643
344	656	668	681	694	706	719	732	744	757	769
345	782	794	807	820	832	845	857	870	882	895
346	908	920	933	945	958	970	983	995	*008	*020
347	54 033	045	058	070	083	095	108	120	133	145
348	158	170	183	195	208	220	233	245	258	270
349	283	295	307	320	332	345	357	370	382	394
350	407	419	432	444	456	469	481	494	506	518
N	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	15	14	13	12
1	1.5	1.4	1.3	1.2
2	3.0	2.8	2.6	2.4
3	4.5	4.2	3.9	3.6
4	6.0	5.6	5.2	4.8
5	7.5	7.0	6.5	6.0
6	9.0	8.4	7.8	7.2
7	10.5	9.8	9.1	8.4
8	12.0	11.2	10.4	9.6
9	13.5	12.6	11.7	10.8

Prop. Parts

N	0	1	2	3	4	5	6	7	8	9
350	54 407	419	432	444	456	469	481	494	506	518
351	531	543	555	568	580	593	605	617	630	642
352	654	667	679	691	704	716	728	741	753	765
353	777	790	802	814	827	839	851	864	876	888
354	900	913	925	937	949	962	974	986	998	*011
355	55 023	035	047	060	072	084	096	108	121	133
356	145	157	169	182	194	206	218	230	242	255
357	267	279	291	303	315	328	340	352	364	376
358	388	400	413	425	437	449	461	473	485	497
359	509	522	534	546	558	570	582	594	606	618
360	630	642	654	666	678	691	703	715	727	739
361	751	763	775	787	799	811	823	835	847	859
362	871	883	895	907	919	931	943	955	967	979
363	991	*003	*015	*027	*038	*050	*062	*074	*086	*098
364	56 110	122	134	146	158	170	182	194	205	217
365	229	241	253	265	277	289	301	312	324	336
366	348	360	372	384	396	407	419	431	443	455
367	467	478	490	502	514	526	538	549	561	573
368	585	597	608	620	632	644	656	667	679	691
369	703	714	726	738	750	761	773	785	797	808
370	820	832	844	855	867	879	891	902	914	926
371	937	949	961	972	984	996	*008	*019	*031	*043
372	57 054	066	078	089	101	113	124	136	148	159
373	171	183	194	206	217	229	241	252	264	276
374	287	299	310	322	334	345	357	368	380	392
375	403	415	426	438	449	461	473	484	496	507
376	519	530	542	553	565	576	588	600	611	623
377	634	646	657	669	680	692	703	715	726	738
378	749	761	772	784	795	807	818	830	841	852
379	864	875	887	898	910	921	933	944	955	967
380	978	990	*001	*013	*024	*035	*047	*058	*070	*081
381	58 092	104	115	127	138	149	161	172	184	195
382	206	218	229	240	252	263	274	286	297	309
383	320	331	343	354	365	377	388	399	410	422
384	433	444	456	467	478	490	501	512	524	535
385	546	557	569	580	591	602	614	625	636	647
386	659	670	681	692	704	715	726	737	749	760
387	771	782	794	805	816	827	838	850	861	872
388	883	894	906	917	928	939	950	961	973	984
389	995	*006	*017	*028	*040	*051	*062	*073	*084	*095
390	59 106	118	129	140	151	162	173	184	195	207
391	218	229	240	251	262	273	284	295	306	318
392	329	340	351	362	373	384	395	406	417	428
393	439	450	461	472	483	494	506	517	528	539
394	550	561	572	583	594	605	616	627	638	649
395	660	671	682	693	704	715	726	737	748	759
396	770	780	791	802	813	824	835	846	857	868
397	879	890	901	912	923	934	945	956	966	977
398	988	999	*010	*021	*032	*043	*054	*065	*076	*086
399	60 097	108	119	130	141	152	163	173	184	195
400	206	217	228	239	249	260	271	282	293	304
N	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	13
1	1.3
2	2.6
3	3.9
4	5.2
5	6.5
6	7.8
7	9.1
8	10.4
9	11.7

	12
1	1.2
2	2.4
3	3.6
4	4.8
5	6.0
6	7.2
7	8.4
8	9.6
9	10.8

	11
1	1.1
2	2.2
3	3.3
4	4.4
5	5.5
6	6.6
7	7.7
8	8.8
9	9.9

	10
1	1.0
2	2.0
3	3.0
4	4.0
5	5.0
6	6.0
7	7.0
8	8.0
9	9.0

Prop. Parts

N	0	1	2	3	4	5	6	7	8	9
400	60 206	217	228	239	249	260	271	282	293	304
401	314	325	336	347	358	369	379	390	401	412
402	423	433	444	455	466	477	487	498	509	520
403	531	541	552	563	574	584	595	606	617	627
404	638	649	660	670	681	692	703	713	724	735
405	746	756	767	778	788	799	810	821	831	842
406	853	863	874	885	895	906	917	927	938	949
407	959	970	981	991	*002	*013	*023	*034	*045	*055
408	61 066	077	087	098	109	119	130	140	151	162
409	172	183	194	204	215	225	236	247	257	268
410	278	289	300	310	321	331	342	352	363	374
411	384	395	405	416	426	437	448	458	469	479
412	490	500	511	521	532	542	553	563	574	584
413	595	606	616	627	637	648	658	669	679	690
414	700	711	721	731	742	752	763	773	784	794
415	805	815	826	836	847	857	868	878	888	899
416	909	920	930	941	951	962	972	982	993	*003
417	62 014	024	034	045	055	066	076	086	097	107
418	118	128	138	149	159	170	180	190	201	211
419	221	232	242	252	263	273	284	294	304	315
420	325	335	346	356	366	377	387	397	408	418
421	428	439	449	459	469	480	490	500	511	521
422	531	542	552	562	572	583	593	603	613	624
423	634	644	655	665	675	685	696	706	716	726
424	737	747	757	767	778	788	798	808	818	829
425	839	849	859	870	880	890	900	910	921	931
426	941	951	961	972	982	992	*002	*012	*022	*033
427	63 043	053	063	073	083	094	104	114	124	134
428	144	155	165	175	185	195	205	215	225	236
429	246	256	266	276	286	296	306	317	327	337
430	347	357	367	377	387	397	407	417	428	438
431	448	458	468	478	488	498	508	518	528	538
432	548	558	568	579	589	599	609	619	629	639
433	649	659	669	679	689	699	709	719	729	739
434	749	759	769	779	789	799	809	819	829	839
435	849	859	869	879	889	899	909	919	929	939
436	949	959	969	979	988	998	*008	*018	*028	*038
437	64 048	058	068	078	088	098	108	118	128	137
438	147	157	167	177	187	197	207	217	227	237
439	246	256	266	276	286	296	306	316	326	335
440	345	355	365	375	385	395	404	414	424	434
441	444	454	464	473	483	493	503	513	523	532
442	542	552	562	572	582	591	601	611	621	631
443	640	650	660	670	680	689	699	709	719	729
444	738	748	758	768	777	787	797	807	816	826
445	836	846	856	865	875	885	895	904	914	924
446	933	943	953	963	972	982	992	*002	*011	*021
447	65 031	040	050	060	070	079	089	099	108	118
448	128	137	147	157	167	176	186	196	205	215
449	225	234	244	254	263	273	283	292	302	312
450	321	331	341	350	360	369	379	389	398	408
N	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	11
1	1.1
2	2.2
3	3.3
4	4.4
5	5.5
6	6.6
7	7.7
8	8.8
9	9.9

	10
1	1.0
2	2.0
3	3.0
4	4.0
5	5.0
6	6.0
7	7.0
8	8.0
9	9.0

	9
1	0.9
2	1.8
3	2.7
4	3.6
5	4.5
6	5.4
7	6.3
8	7.2
9	8.1

Prop. Parts

N	0	1	2	3	4	5	6	7	8	9	Prop. Parts
450	65 321	331	341	350	360	369	379	389	398	408	
451	418	427	437	447	456	466	475	485	495	504	
452	514	523	533	543	552	562	571	581	591	600	
453	610	619	629	639	648	658	667	677	686	696	
454	706	715	725	734	744	753	763	772	782	792	
455	801	811	820	830	839	849	858	868	877	887	
456	896	906	916	925	935	944	954	963	973	982	
457	992	*001	*011	*020	*030	*039	*049	*058	*068	*077	
458	66 087	096	106	115	124	134	143	153	162	172	
459	181	191	200	210	219	229	238	247	257	266	
460	276	285	295	304	314	323	332	342	351	361	
461	370	380	389	398	408	417	427	436	445	455	
462	464	474	483	492	502	511	521	530	539	549	
463	558	567	577	586	596	605	614	624	633	642	
464	652	661	671	680	689	699	708	717	727	736	
465	745	755	764	773	783	792	801	811	820	829	
466	839	848	857	867	876	885	894	904	913	922	
467	932	941	950	960	969	978	987	997	*006	*015	
468	67 025	034	043	052	062	071	080	089	099	108	
469	117	127	136	145	154	164	173	182	191	201	
470	210	219	228	237	247	256	265	274	284	293	
471	302	311	321	330	339	348	357	367	376	385	
472	394	403	413	422	431	440	449	459	468	477	
473	486	495	504	514	523	532	541	550	560	569	
474	578	587	596	605	614	624	633	642	651	660	
475	669	679	688	697	706	715	724	733	742	752	
476	761	770	779	788	797	806	815	825	834	843	
477	852	861	870	879	888	897	906	916	925	934	
478	943	952	961	970	979	988	997	*006	*015	*024	
479	68 034	043	052	061	070	079	088	097	106	115	
480	124	133	142	151	160	169	178	187	196	205	
481	215	224	233	242	251	260	269	278	287	296	
482	305	314	323	332	341	350	359	368	377	386	
483	395	404	413	422	431	440	449	458	467	476	
484	485	494	502	511	520	529	538	547	556	565	
485	574	583	592	601	610	619	628	637	646	655	
486	664	673	681	690	699	708	717	726	735	744	
487	753	762	771	780	789	797	806	815	824	833	
488	842	851	860	869	878	886	895	904	913	922	
489	931	940	949	958	966	975	984	993	*002	*011	
490	69 020	028	037	046	055	064	073	082	090	099	
491	108	117	126	135	144	152	161	170	179	188	
492	197	205	214	223	232	241	249	258	267	276	
493	285	294	302	311	320	329	338	346	355	364	
494	373	381	390	399	408	417	425	434	443	452	
495	461	469	478	487	496	504	513	522	531	539	
496	548	557	566	574	583	592	601	609	618	627	
497	636	644	653	662	671	679	688	697	705	714	
498	723	732	740	749	758	767	775	784	793	801	
499	810	819	827	836	845	854	862	871	880	888	
500	897	906	914	923	932	940	949	958	966	975	
N	0	1	2	3	4	5	6	7	8	9	Prop. Parts

Prop. Parts

	10
1	1.0
2	2.0
3	3.0
4	4.0
5	5.0
6	6.0
7	7.0
8	8.0
9	9.0

	9
1	0.9
2	1.8
3	2.7
4	3.6
5	4.5
6	5.4
7	6.3
8	7.2
9	8.1

	8
1	0.8
2	1.6
3	2.4
4	3.2
5	4.0
6	4.8
7	5.6
8	6.4
9	7.2

450 — Five-Place Common Logarithms — 500

N	0	1	2	3	4	5	6	7	8	9
500	69 897	906	914	923	932	940	949	958	966	975
501	984	992	*001	*010	*018	*027	*036	*044	*053	*062
502	70 070	079	088	096	105	114	122	131	140	148
503	157	165	174	183	191	200	209	217	226	234
504	243	252	260	269	278	286	295	303	312	321
505	329	338	346	355	364	372	381	389	398	406
506	415	424	432	441	449	458	467	475	484	492
507	501	509	518	526	535	544	552	561	569	578
508	586	595	603	612	621	629	638	646	655	663
509	672	680	689	697	706	714	723	731	740	749
510	757	766	774	783	791	800	808	817	825	834
511	842	851	859	868	876	885	893	902	910	919
512	927	935	944	952	961	969	978	986	995	*003
513	71 012	020	029	037	046	054	063	071	079	088
514	096	105	113	122	130	139	147	155	164	172
515	181	189	198	206	214	223	231	240	248	257
516	265	273	282	290	299	307	315	324	332	341
517	349	357	366	374	383	391	399	408	416	425
518	433	441	450	458	466	475	483	492	500	508
519	517	525	533	542	550	559	567	575	584	592
520	600	609	617	625	634	642	650	659	667	675
521	684	692	700	709	717	725	734	742	750	759
522	767	775	784	792	800	809	817	825	834	842
523	850	858	867	875	883	892	900	908	917	925
524	933	941	950	958	966	975	983	991	999	*008
525	72 016	024	032	041	049	057	066	074	082	090
526	099	107	115	123	132	140	148	156	165	173
527	181	189	198	206	214	222	230	239	247	255
528	263	272	280	288	296	304	313	321	329	337
529	346	354	362	370	378	387	395	403	411	419
530	428	436	444	452	460	469	477	485	493	501
531	509	518	526	534	542	550	558	567	575	583
532	591	599	607	616	624	632	640	648	656	665
533	673	681	689	697	705	713	722	730	738	746
534	754	762	770	779	787	795	803	811	819	827
535	835	843	852	860	868	876	884	892	900	908
536	916	925	933	941	949	957	965	973	981	989
537	997	*006	*014	*022	*030	*038	*046	*054	*062	*070
538	73 078	086	094	102	111	119	127	135	143	151
539	159	167	175	183	191	199	207	215	223	231
540	239	247	255	263	272	280	288	296	304	312
541	320	328	336	344	352	360	368	376	384	392
542	400	408	416	424	432	440	448	456	464	472
543	480	488	496	504	512	520	528	536	544	552
544	560	568	576	584	592	600	608	616	624	632
545	640	648	656	664	672	679	687	695	703	711
546	719	727	735	743	751	759	767	775	783	791
547	799	807	815	823	830	838	846	854	862	870
548	878	886	894	902	910	918	926	933	941	949
549	957	965	973	981	989	997	*005	*013	*020	*028
550	74 036	044	052	060	068	076	084	092	099	107
N	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	9
1	0.9
2	1.8
3	2.7
4	3.6
5	4.5
6	5.4
7	6.3
8	7.2
9	8.1

	8
1	0.8
2	1.6
3	2.4
4	3.2
5	4.0
6	4.8
7	5.6
8	6.4
9	7.2

	7
1	0.7
2	1.4
3	2.1
4	2.8
5	3.5
6	4.2
7	4.9
8	5.6
9	6.3

Prop. Parts

N	0	1	2	3	4	5	6	7	8	9
550	74 036	044	052	060	068	076	084	092	099	107
551	115	123	131	139	147	155	162	170	178	186
552	194	202	210	218	225	233	241	249	257	265
553	273	280	288	296	304	312	320	327	335	343
554	351	359	367	374	382	390	398	406	414	421
555	429	437	445	453	461	468	476	484	492	500
556	507	515	523	531	539	547	554	562	570	578
557	586	593	601	609	617	624	632	640	648	656
558	663	671	679	687	695	702	710	718	726	733
559	741	749	757	764	772	780	788	796	803	811
560	819	827	834	842	850	858	865	873	881	889
561	896	904	912	920	927	935	943	950	958	966
562	974	981	989	997	*005	*012	*020	*028	*035	*043
563	75 051	059	066	074	082	089	097	105	113	120
564	128	136	143	151	159	166	174	182	189	197
565	205	213	220	228	236	243	251	259	266	274
566	282	289	297	305	312	320	328	335	343	351
567	358	366	374	381	389	397	404	412	420	427
568	435	442	450	458	465	473	481	488	496	504
569	511	519	526	534	542	549	557	565	572	580
570	587	595	603	610	618	626	633	641	648	656
571	664	671	679	686	694	702	709	717	724	732
572	740	747	755	762	770	778	785	793	800	808
573	815	823	831	838	846	853	861	868	876	884
574	891	899	906	914	921	929	937	944	952	959
575	967	974	982	989	997	*005	*012	*020	*027	*035
576	76 042	050	057	065	072	080	087	095	103	110
577	118	125	133	140	148	155	163	170	178	185
578	193	200	208	215	223	230	238	245	253	260
579	268	275	283	290	298	305	313	320	328	335
580	343	350	358	365	373	380	388	395	403	410
581	418	425	433	440	448	455	462	470	477	485
582	492	500	507	515	522	530	537	545	552	559
583	567	574	582	589	597	604	612	619	626	634
584	641	649	656	664	671	678	686	693	701	708
585	716	723	730	738	745	753	760	768	775	782
586	790	797	805	812	819	827	834	842	849	856
587	864	871	879	886	893	901	908	916	923	930
588	938	945	953	960	967	975	982	989	997	*004
589	77 012	019	026	034	041	048	056	063	070	078
590	085	093	100	107	115	122	129	137	144	151
591	159	166	173	181	188	195	203	210	217	225
592	232	240	247	254	262	269	276	283	291	298
593	305	313	320	327	335	342	349	357	364	371
594	379	386	393	401	408	415	422	430	437	444
595	452	459	466	474	481	488	495	503	510	517
596	525	532	539	546	554	561	568	576	583	590
597	597	605	612	619	627	634	641	648	656	663
598	670	677	685	692	699	706	714	721	728	735
599	743	750	757	764	772	779	786	793	801	808
600	815	822	830	837	844	851	859	866	873	880
N	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	8
1	0.8
2	1.6
3	2.4
4	3.2
5	4.0
6	4.8
7	5.6
8	6.4
9	7.2

	7
1	0.7
2	1.4
3	2.1
4	2.8
5	3.5
6	4.2
7	4.9
8	5.6
9	6.3

Prop. Parts

N	0	1	2	3	4	5	6	7	8	9
600	77 815	822	830	837	844	851	859	866	873	880
601	887	895	902	909	916	924	931	938	945	952
602	960	967	974	981	988	996	*003	*010	*017	*025
603	78 032	039	046	053	061	068	075	082	089	097
604	104	111	118	125	132	140	147	154	161	168
605	176	183	190	197	204	211	219	226	233	240
606	247	254	262	269	276	283	290	297	305	312
607	319	326	333	340	347	355	362	369	376	383
608	390	398	405	412	419	426	433	440	447	455
609	462	469	476	483	490	497	504	512	519	526
610	533	540	547	554	561	569	576	583	590	597
611	604	611	618	625	633	640	647	654	661	668
612	675	682	689	696	704	711	718	725	732	739
613	746	753	760	767	774	781	789	796	803	810
614	817	824	831	838	845	852	859	866	873	880
615	888	895	902	909	916	923	930	937	944	951
616	958	965	972	979	986	993	*000	*007	*014	*021
617	79 029	036	043	050	057	064	071	078	085	092
618	099	106	113	120	127	134	141	148	155	162
619	169	176	183	190	197	204	211	218	225	232
620	239	246	253	260	267	274	281	288	295	302
621	309	316	323	330	337	344	351	358	365	372
622	379	386	393	400	407	414	421	428	435	442
623	449	456	463	470	477	484	491	498	505	511
624	518	525	532	539	546	553	560	567	574	581
625	588	595	602	609	616	623	630	637	644	650
626	657	664	671	678	685	692	699	706	713	720
627	727	734	741	748	754	761	768	775	782	789
628	796	803	810	817	824	831	837	844	851	858
629	865	872	879	886	893	900	906	913	920	927
630	934	941	948	955	962	969	975	982	989	996
631	80 003	010	017	024	030	037	044	051	058	065
632	072	079	085	092	099	106	113	120	127	134
633	140	147	154	161	168	175	182	188	195	202
634	209	216	223	229	236	243	250	257	264	271
635	277	284	291	298	305	312	318	325	332	339
636	346	353	359	366	373	380	387	393	400	407
637	414	421	428	434	441	448	455	462	468	475
638	482	489	496	502	509	516	523	530	536	543
639	550	557	564	570	577	584	591	598	604	611
640	618	625	632	638	645	652	659	665	672	679
641	686	693	699	706	713	720	726	733	740	747
642	754	760	767	774	781	787	794	801	808	814
643	821	828	835	841	848	855	862	868	875	882
644	889	895	902	909	916	922	929	936	943	949
645	956	963	969	976	983	990	996	*003	*010	*017
646	81 023	030	037	043	050	057	064	070	077	084
647	090	097	104	111	117	124	131	137	144	151
648	158	164	171	178	184	191	198	204	211	218
649	224	231	238	245	251	258	265	271	278	285
650	291	298	305	311	318	325	331	338	345	351
N	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	8
1	0.8
2	1.6
3	2.4
4	3.2
5	4.0
6	4.8
7	5.6
8	6.4
9	7.2

	7
1	0.7
2	1.4
3	2.1
4	2.8
5	3.5
6	4.2
7	4.9
8	5.6
9	6.3

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

Prop. Parts

N	0	1	2	3	4	5	6	7	8	9
650	81 291	298	305	311	318	325	331	338	345	351
651	358	365	371	378	385	391	398	405	411	418
652	425	431	438	445	451	458	465	471	478	485
653	491	498	505	511	518	525	531	538	544	551
654	558	564	571	578	584	591	598	604	611	617
655	624	631	637	644	651	657	664	671	677	684
656	690	697	704	710	717	723	730	737	743	750
657	757	763	770	776	783	790	796	803	809	816
658	823	829	836	842	849	856	862	869	875	882
659	889	895	902	908	915	921	928	935	941	948
660	954	961	968	974	981	987	994	*000	*007	*014
661	82 020	027	033	040	046	053	060	066	073	079
662	086	092	099	105	112	119	125	132	138	145
663	151	158	164	171	178	184	191	197	204	210
664	217	223	230	236	243	249	256	263	269	276
665	282	289	295	302	308	315	321	328	334	341
666	347	354	360	367	373	380	387	393	400	406
667	413	419	426	432	439	445	452	458	465	471
668	478	484	491	497	504	510	517	523	530	536
669	543	549	556	562	569	575	582	588	595	601
670	607	614	620	627	633	640	646	653	659	666
671	672	679	685	692	698	705	711	718	724	730
672	737	743	750	756	763	769	776	782	789	795
673	802	808	814	821	827	834	840	847	853	860
674	866	872	879	885	892	898	905	911	918	924
675	930	937	943	950	956	963	969	975	982	988
676	995	*001	*008	*014	*020	*027	*033	*040	*046	*052
677	83 059	065	072	078	085	091	097	104	110	117
678	123	129	136	142	149	155	161	168	174	181
679	187	193	200	206	213	219	225	232	238	245
680	251	257	264	270	276	283	289	296	302	308
681	315	321	327	334	340	347	353	359	366	372
682	378	385	391	398	404	410	417	423	429	436
683	442	448	455	461	467	474	480	487	493	499
684	506	512	518	525	531	537	544	550	556	563
685	569	575	582	588	594	601	607	613	620	626
686	632	639	645	651	658	664	670	677	683	689
687	696	702	708	715	721	727	734	740	746	753
688	759	765	771	778	784	790	797	803	809	816
689	822	828	835	841	847	853	860	866	872	879
690	885	891	897	904	910	916	923	929	935	942
691	948	954	960	967	973	979	985	992	998	*004
692	84 011	017	023	029	036	042	048	055	061	067
693	073	080	086	092	098	105	111	117	123	130
694	136	142	148	155	161	167	173	180	186	192
695	198	205	211	217	223	230	236	242	248	255
696	261	267	273	280	286	292	298	305	311	317
697	323	330	336	342	348	354	361	367	373	379
698	386	392	398	404	410	417	423	429	435	442
699	448	454	460	466	473	479	485	491	497	504
700	510	516	522	528	535	541	547	553	559	566
N	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	7
1	0.7
2	1.4
3	2.1
4	2.8
5	3.5
6	4.2
7	4.9
8	5.6
9	6.3

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

Prop. Parts

N	0	1	2	3	4	5	6	7	8	9
700	84 510	516	522	528	535	541	547	553	559	566
701	572	578	584	590	597	603	609	615	621	628
702	634	640	646	652	658	665	671	677	683	689
703	696	702	708	714	720	726	733	739	745	751
704	757	763	770	776	782	788	794	800	807	813
705	819	825	831	837	844	850	856	862	868	874
706	880	887	893	899	905	911	917	924	930	936
707	942	948	954	960	967	973	979	985	991	997
708	85 003	009	016	022	028	034	040	046	052	058
709	065	071	077	083	089	095	101	107	114	120
710	126	132	138	144	150	156	163	169	175	181
711	187	193	199	205	211	217	224	230	236	242
712	248	254	260	266	272	278	285	291	297	303
713	309	315	321	327	333	339	345	352	358	364
714	370	376	382	388	394	400	406	412	418	425
715	431	437	443	449	455	461	467	473	479	485
716	491	497	503	509	516	522	528	534	540	546
717	552	558	564	570	576	582	588	594	600	606
718	612	618	625	631	637	643	649	655	661	667
719	673	679	685	691	697	703	709	715	721	727
720	733	739	745	751	757	763	769	775	781	788
721	794	800	806	812	818	824	830	836	842	848
722	854	860	866	872	878	884	890	896	902	908
723	914	920	926	932	938	944	950	956	962	968
724	974	980	986	992	998	*004	*010	*016	*022	*028
725	86 034	040	046	052	058	064	070	076	082	088
726	094	100	106	112	118	124	130	136	141	147
727	153	159	165	171	177	183	189	195	201	207
728	213	219	225	231	237	243	249	255	261	267
729	273	279	285	291	297	303	308	314	320	326
730	332	338	344	350	356	362	368	374	380	386
731	392	398	404	410	415	421	427	433	439	445
732	451	457	463	469	475	481	487	493	499	504
733	510	516	522	528	534	540	546	552	558	564
734	570	576	581	587	593	599	605	611	617	623
735	629	635	641	646	652	658	664	670	676	682
736	688	694	700	705	711	717	723	729	735	741
737	747	753	759	764	770	776	782	788	794	800
738	806	812	817	823	829	835	841	847	853	859
739	864	870	876	882	888	894	900	906	911	917
740	923	929	935	941	947	953	958	964	970	976
741	982	988	994	999	*005	*011	*017	*023	*029	*035
742	87 040	046	052	058	064	070	075	081	087	093
743	099	105	111	116	122	128	134	140	146	151
744	157	163	169	175	181	186	192	198	204	210
745	216	221	227	233	239	245	251	256	262	268
746	274	280	286	291	297	303	309	315	320	326
747	332	338	344	349	355	361	367	373	379	384
748	390	396	402	408	413	419	425	431	437	442
749	448	454	460	466	471	477	483	489	495	500
750	506	512	518	523	529	535	541	547	552	558
N	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	7
1	0.7
2	1.4
3	2.1
4	2.8
5	3.5
6	4.2
7	4.9
8	5.6
9	6.3

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

Prop. Parts

N	0	1	2	3	4	5	6	7	8	9
750	87 506	512	518	523	529	535	541	547	552	558
751	564	570	576	581	587	593	599	604	610	616
752	622	628	633	639	645	651	656	662	668	674
753	679	685	691	697	703	708	714	720	726	731
754	737	743	749	754	760	766	772	777	783	789
755	795	800	806	812	818	823	829	835	841	846
756	852	858	864	869	875	881	887	892	898	904
757	910	915	921	927	933	938	944	950	955	961
758	967	973	978	984	990	996	*001	*007	*013	*018
759	88 024	030	036	041	047	053	058	064	070	076
760	081	087	093	098	104	110	116	121	127	133
761	138	144	150	156	161	167	173	178	184	190
762	195	201	207	213	218	224	230	235	241	247
763	252	258	264	270	275	281	287	292	298	304
764	309	315	321	326	332	338	343	349	355	360
765	366	372	377	383	389	395	400	406	412	417
766	423	429	434	440	446	451	457	463	468	474
767	480	485	491	497	502	508	513	519	525	530
768	536	542	547	553	559	564	570	576	581	587
769	593	598	604	610	615	621	627	632	638	643
770	649	655	660	666	672	677	683	689	694	700
771	705	711	717	722	728	734	739	745	750	756
772	762	767	773	779	784	790	795	801	807	812
773	818	824	829	835	840	846	852	857	863	868
774	874	880	885	891	897	902	908	913	919	925
775	930	936	941	947	953	958	964	969	975	981
776	986	992	997	*003	*009	*014	*020	*025	*031	*037
777	89 042	048	053	059	064	070	076	081	087	092
778	098	104	109	115	120	126	131	137	143	148
779	154	159	165	170	176	182	187	193	198	204
780	209	215	221	226	232	237	243	248	254	260
781	265	271	276	282	287	293	298	304	310	315
782	321	326	332	337	343	348	354	360	365	371
783	376	382	387	393	398	404	409	415	421	426
784	432	437	443	448	454	459	465	470	476	481
785	487	492	498	504	509	515	520	526	531	537
786	542	548	553	559	564	570	575	581	586	592
787	597	603	609	614	620	625	631	636	642	647
788	653	658	664	669	675	680	686	691	697	702
789	708	713	719	724	730	735	741	746	752	757
790	763	768	774	779	785	790	796	801	807	812
791	818	823	829	834	840	845	851	856	862	867
792	873	878	883	889	894	900	905	911	916	922
793	927	933	938	944	949	955	960	966	971	977
794	982	988	993	998	*004	*009	*015	*020	*026	*031
795	90 037	042	048	053	059	064	069	075	080	086
796	091	097	102	108	113	119	124	129	135	140
797	146	151	157	162	168	173	179	184	189	195
798	200	206	211	217	222	227	233	238	244	249
799	255	260	266	271	276	282	287	293	298	304
800	309	314	320	325	331	336	342	347	352	358
N	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

Prop. Parts

750 — Five-Place Common Logarithms — 800

N	0	1	2	3	4	5	6	7	8	9
800	90 309	314	320	325	331	336	342	347	352	358
801	363	369	374	380	385	390	396	401	407	412
802	417	423	428	434	439	445	450	455	461	466
803	472	477	482	488	493	499	504	509	515	520
804	526	531	536	542	547	553	558	563	569	574
805	580	585	590	596	601	607	612	617	623	628
806	634	639	644	650	655	660	666	671	677	682
807	687	693	698	703	709	714	720	725	730	736
808	741	747	752	757	763	768	773	779	784	789
809	795	800	806	811	816	822	827	832	838	843
810	849	854	859	865	870	875	881	886	891	897
811	902	907	913	918	924	929	934	940	945	950
812	956	961	966	972	977	982	988	993	998	*004
813	91 009	014	020	025	030	036	041	046	052	057
814	062	068	073	078	084	089	094	100	105	110
815	116	121	126	132	137	142	148	153	158	164
816	169	174	180	185	190	196	201	206	212	217
817	222	228	233	238	243	249	254	259	265	270
818	275	281	286	291	297	302	307	312	318	323
819	328	334	339	344	350	355	360	365	371	376
820	381	387	392	397	403	408	413	418	424	429
821	434	440	445	450	455	461	466	471	477	482
822	487	492	498	503	508	514	519	524	529	535
823	540	545	551	556	561	566	572	577	582	587
824	593	598	603	609	614	619	624	630	635	640
825	645	651	656	661	666	672	677	682	687	693
826	698	703	709	714	719	724	730	735	740	745
827	751	756	761	766	772	777	782	787	793	798
828	803	808	814	819	824	829	834	840	845	850
829	855	861	866	871	876	882	887	892	897	903
830	908	913	918	924	929	934	939	944	950	955
831	960	965	971	976	981	986	991	997	*002	*007
832	92 012	018	023	028	033	038	044	049	054	059
833	065	070	075	080	085	091	096	101	106	111
834	117	122	127	132	137	143	148	153	158	163
835	169	174	179	184	189	195	200	205	210	215
836	221	226	231	236	241	247	252	257	262	267
837	273	278	283	288	293	298	304	309	314	319
838	324	330	335	340	345	350	355	361	366	371
839	376	381	387	392	397	402	407	412	418	423
840	428	433	438	443	449	454	459	464	469	474
841	480	485	490	495	500	505	511	516	521	526
842	531	536	542	547	552	557	562	567	572	578
843	583	588	593	598	603	609	614	619	624	629
844	634	639	645	650	655	660	665	670	675	681
845	686	691	696	701	706	711	716	722	727	732
846	737	742	747	752	758	763	768	773	778	783
847	788	793	799	804	809	814	819	824	829	834
848	840	845	850	855	860	865	870	875	881	886
849	891	896	901	906	911	916	921	927	932	937
850	942	947	952	957	962	967	973	978	983	988
N	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

Prop. Parts

N	0	1	2	3	4	5	6	7	8	9
850	92 942	947	952	957	962	967	973	978	983	988
851	993	998	*003	*008	*013	*018	*024	*029	*034	*039
852	93 044	049	054	059	064	069	075	080	085	090
853	095	100	105	110	115	120	125	131	136	141
854	146	151	156	161	166	171	176	181	186	192
855	197	202	207	212	217	222	227	232	237	242
856	247	252	258	263	268	273	278	283	288	293
857	298	303	308	313	318	323	328	334	339	344
858	349	354	359	364	369	374	379	384	389	394
859	399	404	409	414	420	425	430	435	440	445
860	450	455	460	465	470	475	480	485	490	495
861	500	505	510	515	520	526	531	536	541	546
862	551	556	561	566	571	576	581	586	591	596
863	601	606	611	616	621	626	631	636	641	646
864	651	656	661	666	671	676	682	687	692	697
865	702	707	712	717	722	727	732	737	742	747
866	752	757	762	767	772	777	782	787	792	797
867	802	807	812	817	822	827	832	837	842	847
868	852	857	862	867	872	877	882	887	892	897
869	902	907	912	917	922	927	932	937	942	947
870	952	957	962	967	972	977	982	987	992	997
871	94 002	007	012	017	022	027	032	037	042	047
872	052	057	062	067	072	077	082	086	091	096
873	101	106	111	116	121	126	131	136	141	146
874	151	156	161	166	171	176	181	186	191	196
875	201	206	211	216	221	226	231	236	240	245
876	250	255	260	265	270	275	280	285	290	295
877	300	305	310	315	320	325	330	335	340	345
878	349	354	359	364	369	374	379	384	389	394
879	399	404	409	414	419	424	429	433	438	443
880	448	453	458	463	468	473	478	483	488	493
881	498	503	507	512	517	522	527	532	537	542
882	547	552	557	562	567	571	576	581	586	591
883	596	601	606	611	616	621	626	630	635	640
884	645	650	655	660	665	670	675	680	685	689
885	694	699	704	709	714	719	724	729	734	738
886	743	748	753	758	763	768	773	778	783	787
887	792	797	802	807	812	817	822	827	832	836
888	841	846	851	856	861	866	871	876	880	885
889	890	895	900	905	910	915	919	924	929	934
890	939	944	949	954	959	963	968	973	978	983
891	988	993	998	*002	*007	*012	*017	*022	*027	*032
892	95 036	041	046	051	056	061	066	071	075	080
893	085	090	095	100	105	109	114	119	124	129
894	134	139	143	148	153	158	163	168	173	177
895	182	187	192	197	202	207	211	216	221	226
896	231	236	240	245	250	255	260	265	270	274
897	279	284	289	294	299	303	308	313	318	323
898	328	332	337	342	347	352	357	361	366	371
899	376	381	386	390	395	400	405	410	415	419
900	424	429	434	439	444	448	453	458	463	468
N	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

	4
1	0.4
2	0.8
3	1.2
4	1.6
5	2.0
6	2.4
7	2.8
8	3.2
9	3.6

Prop. Parts

N	0	1	2	3	4	5	6	7	8	9
900	95 424	429	434	439	444	448	453	458	463	468
901	472	477	482	487	492	497	501	506	511	516
902	521	525	530	535	540	545	550	554	559	564
903	569	574	578	583	588	593	598	602	607	612
904	617	622	626	631	636	641	646	650	655	660
905	665	670	674	679	684	689	694	698	703	708
906	713	718	722	727	732	737	742	746	751	756
907	761	766	770	775	780	785	789	794	799	804
908	809	813	818	823	828	832	837	842	847	852
909	856	861	866	871	875	880	885	890	895	899
910	904	909	914	918	923	928	933	938	942	947
911	952	957	961	966	971	976	980	985	990	995
912	999	*004	*009	*014	*019	*023	*028	*033	*038	*042
913	96 047	052	057	061	066	071	076	080	085	090
914	095	099	104	109	114	118	123	128	133	137
915	142	147	152	156	161	166	171	175	180	185
916	190	194	199	204	209	213	218	223	227	232
917	237	242	246	251	256	261	265	270	275	280
918	284	289	294	298	303	308	313	317	322	327
919	332	336	341	346	350	355	360	365	369	374
920	379	384	388	393	398	402	407	412	417	421
921	426	431	435	440	445	450	454	459	464	468
922	473	478	483	487	492	497	501	506	511	515
923	520	525	530	534	539	544	548	553	558	562
924	567	572	577	581	586	591	595	600	605	609
925	614	619	624	628	633	638	642	647	652	656
926	661	666	670	675	680	685	689	694	699	703
927	708	713	717	722	727	731	736	741	745	750
928	755	759	764	769	774	778	783	788	792	797
929	802	806	811	816	820	825	830	834	839	844
930	848	853	858	862	867	872	876	881	886	890
931	895	900	904	909	914	918	923	928	932	937
932	942	946	951	956	960	965	970	974	979	984
933	988	993	997	*002	*007	*011	*016	*021	*025	*030
934	97 035	039	044	049	053	058	063	067	072	077
935	081	086	090	095	100	104	109	114	118	123
936	128	132	137	142	146	151	155	160	165	169
937	174	179	183	188	192	197	202	206	211	216
938	220	225	230	234	239	243	248	253	257	262
939	267	271	276	280	285	290	294	299	304	308
940	313	317	322	327	331	336	340	345	350	354
941	359	364	368	373	377	382	387	391	396	400
942	405	410	414	419	424	428	433	437	442	447
943	451	456	460	465	470	474	479	483	488	493
944	497	502	506	511	516	520	525	529	534	539
945	543	548	552	557	562	566	571	575	580	585
946	589	594	598	603	607	612	617	621	626	630
947	635	640	644	649	653	658	663	667	672	676
948	681	685	690	695	699	704	708	713	717	722
949	727	731	736	740	745	749	754	759	763	768
950	772	777	782	786	791	795	800	804	809	813
N	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

	4
1	0.4
2	0.8
3	1.2
4	1.6
5	2.0
6	2.4
7	2.8
8	3.2
9	3.6

Prop. Parts

N	0	1	2	3	4	5	6	7	8	9
950	97 772	777	782	786	791	795	800	804	809	813
951	818	823	827	832	836	841	845	850	855	859
952	864	868	873	877	882	886	891	896	900	905
953	909	914	918	923	928	932	937	941	946	950
954	955	959	964	968	973	978	982	987	991	996
955	98 000	005	009	014	019	023	028	032	037	041
956	046	050	055	059	064	068	073	078	082	087
957	091	096	100	105	109	114	118	123	127	132
958	137	141	146	150	155	159	164	168	173	177
959	182	186	191	195	200	204	209	214	218	223
960	227	232	236	241	245	250	254	259	263	268
961	272	277	281	286	290	295	299	304	308	313
962	318	322	327	331	336	340	345	349	354	358
963	363	367	372	376	381	385	390	394	399	403
964	408	412	417	421	426	430	435	439	444	448
965	453	457	462	466	471	475	480	484	489	493
966	498	502	507	511	516	520	525	529	534	538
967	543	547	552	556	561	565	570	574	579	583
968	588	592	597	601	605	610	614	619	623	628
969	632	637	641	646	650	655	659	664	668	673
970	677	682	686	691	695	700	704	709	713	717
971	722	726	731	735	740	744	749	753	758	762
972	767	771	776	780	784	789	793	798	802	807
973	811	816	820	825	829	834	838	843	847	851
974	856	860	865	869	874	878	883	887	892	896
975	900	905	909	914	918	923	927	932	936	941
976	945	949	954	958	963	967	972	976	981	985
977	989	994	998	*003	*007	*012	*016	*021	*025	*029
978	99 034	038	043	047	052	056	061	065	069	074
979	078	083	087	092	096	100	105	109	114	118
980	123	127	131	136	140	145	149	154	158	162
981	167	171	176	180	185	189	193	198	202	207
982	211	216	220	224	229	233	238	242	247	251
983	255	260	264	269	273	277	282	286	291	295
984	300	304	308	313	317	322	326	330	335	339
985	344	348	352	357	361	366	370	374	379	383
986	388	392	396	401	405	410	414	419	423	427
987	432	436	441	445	449	454	458	463	467	471
988	476	480	484	489	493	498	502	506	511	515
989	520	524	528	533	537	542	546	550	555	559
990	564	568	572	577	581	585	590	594	599	603
991	607	612	616	621	625	629	634	638	642	647
992	651	656	660	664	669	673	677	682	686	691
993	695	699	704	708	712	717	721	726	730	734
994	739	743	747	752	756	760	765	769	774	778
995	782	787	791	795	800	804	808	813	817	822
996	826	830	835	839	843	848	852	856	861	865
997	870	874	878	883	887	891	896	900	904	909
998	913	917	922	926	930	935	939	944	948	952
999	957	961	965	970	974	978	983	987	991	996
1000	00 000	004	009	013	017	022	026	030	035	039
N	0	1	2	3	4	5	6	7	8	9

Prop. Parts

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

	4
1	0.4
2	0.8
3	1.2
4	1.6
5	2.0
6	2.4
7	2.8
8	3.2
9	3.6

Prop. Parts

TABLE III
Natural Logarithms

N	0	1	2	3	4	5	6	7	8	9
1.0	0.0 000	100	198	296	392	488	583	677	770	862
1.1	0.0 953	*044	*133	*222	*310	*398	*484	*570	*655	*740
1.2	0.1 823	906	989	*070	*151	*231	*311	*390	*469	*546
1.3	0.2 624	700	776	852	927	*001	*075	*148	*221	*293
1.4	0.3 365	436	507	577	646	716	784	853	920	988
1.5	0.4 055	121	187	253	318	383	447	511	574	637
1.6	0.4 700	762	824	886	947	*008	*068	*128	*188	*247
1.7	0.5 306	365	423	481	539	596	653	710	766	822
1.8	0.5 878	933	988	*043	*098	*152	*206	*259	*313	*366
1.9	0.6 419	471	523	575	627	678	729	780	831	881
2.0	0.6 931	981	*031	*080	*129	*178	*227	*275	*324	*372
2.1	0.7 419	467	514	561	608	655	701	747	793	839
2.2	0.7 885	930	975	*020	*065	*109	*154	*198	*242	*286
2.3	0.8 329	372	416	459	502	544	587	629	671	713
2.4	0.8 755	796	838	879	920	961	*002	*042	*083	*123
2.5	0.9 163	203	243	282	322	361	400	439	478	517
2.6	555	594	632	670	708	746	783	821	858	895
2.7	0.9 933	969	*006	*043	*080	*116	*152	*188	*225	*260
2.8	1.0 296	332	367	403	438	473	508	543	578	613
2.9	647	682	716	750	784	818	852	886	919	953
3.0	1.0 986	*019	*053	*086	*119	*151	*184	*217	*249	*282
3.1	1.1 314	346	378	410	442	474	506	537	569	600
3.2	632	663	694	725	756	787	817	848	878	909
3.3	1.1 939	969	*000	*030	*060	*090	*119	*149	*179	*208
3.4	1.2 238	267	296	326	355	384	413	442	470	499
3.5	528	556	585	613	641	669	698	726	754	782
3.6	1.2 809	837	865	892	920	947	975	*002	*029	*056
3.7	1.3 083	110	137	164	191	218	244	271	297	324
3.8	350	376	403	429	455	481	507	533	558	584
3.9	610	635	661	686	712	737	762	788	813	838
4.0	1.3 863	888	913	938	962	987	*012	*036	*061	*085
4.1	1.4 110	134	159	183	207	231	255	279	303	327
4.2	351	375	398	422	446	469	493	516	540	563
4.3	586	609	633	656	679	702	725	748	770	793
4.4	1.4 816	839	861	884	907	929	951	974	996	*019
4.5	1.5 041	063	085	107	129	151	173	195	217	239
4.6	261	282	304	326	347	369	390	412	433	454
4.7	476	497	518	539	560	581	602	623	644	665
4.8	686	707	728	748	769	790	810	831	851	872
4.9	1.5 892	913	933	953	974	994	*014	*034	*054	*074
5.0	1.6 094	114	134	154	174	194	214	233	253	273
5.1	292	312	332	351	371	390	409	429	448	467
5.2	487	506	525	544	563	582	601	620	639	658
5.3	677	696	715	734	752	771	790	808	827	845
5.4	1.6 864	882	901	919	938	956	974	993	*011	*029
5.5	1.7 047	066	084	102	120	138	156	174	192	210
5.6	228	246	263	281	299	317	334	352	370	387
5.7	405	422	440	457	475	492	509	527	544	561
5.8	579	596	613	630	647	664	681	699	716	733
5.9	750	766	783	800	817	834	851	867	884	901
6.0	1.7 918	934	951	967	984	*001	*017	*034	*050	*066
N	0	1	2	3	4	5	6	7	8	9

Natural Logarithms

N	0	1	2	3	4	5	6	7	8	9
6.0	1.7 918	934	951	967	984	*001	*017	*034	*050	*066
6.1	1.8 083	099	116	132	148	165	181	197	213	229
6.2	245	262	278	294	310	326	342	358	374	390
6.3	405	421	437	453	469	485	500	516	532	547
6.4	563	579	594	610	625	641	656	672	687	703
6.5	718	733	749	764	779	795	810	825	840	856
6.6	1.8 871	886	901	916	931	946	961	976	991	*006
6.7	1.9 021	036	051	066	081	095	110	125	140	155
6.8	169	184	199	213	228	242	257	272	286	301
6.9	315	330	344	359	373	387	402	416	430	445
7.0	459	473	488	502	516	530	544	559	573	587
7.1	601	615	629	643	657	671	685	699	713	727
7.2	741	755	769	782	796	810	824	838	851	865
7.3	1.9 879	892	906	920	933	947	961	974	988	*001
7.4	2.0 015	028	042	055	069	082	096	109	122	136
7.5	149	162	176	189	202	215	229	242	255	268
7.6	281	295	308	321	334	347	360	373	386	399
7.7	412	425	438	451	464	477	490	503	516	528
7.8	541	554	567	580	592	605	618	631	643	656
7.9	669	681	694	707	719	732	744	757	769	782
8.0	794	807	819	832	844	857	869	882	894	906
8.1	2.0 919	931	943	956	968	980	992	*005	*017	*029
8.2	2.1 041	054	066	078	090	102	114	126	138	150
8.3	163	175	187	199	211	223	235	247	258	270
8.4	282	294	306	318	330	342	353	365	377	389
8.5	401	412	424	436	448	459	471	483	494	506
8.6	518	529	541	552	564	576	587	599	610	622
8.7	633	645	656	668	679	691	702	713	725	736
8.8	748	759	770	782	793	804	815	827	838	849
8.9	861	872	883	894	905	917	928	939	950	961
9.0	2.1 972	983	994	*006	*017	*028	*039	*050	*061	*072
9.1	2.2 083	094	105	116	127	138	148	159	170	181
9.2	192	203	214	225	235	246	257	268	279	289
9.3	300	311	322	332	343	354	364	375	386	396
9.4	407	418	428	439	450	460	471	481	492	502
9.5	513	523	534	544	555	565	576	586	597	607
9.6	618	628	638	649	659	670	680	690	701	711
9.7	721	732	742	752	762	773	783	793	803	814
9.8	824	834	844	854	865	875	885	895	905	915
9.9	2.2 925	935	946	956	966	976	986	996	*006	*016
10.0	2.3 026	036	046	056	066	076	086	096	106	115
N	0	1	2	3	4	5	6	7	8	9

Natural Logarithms

To find ln P when P is above or below the range of this table, write P in the form $N \cdot 10^m$, where N lies within the range of the table and m is a positive or negative integer; then use

$$\ln P = \ln(N \cdot 10^m) = \ln N + m \ln 10.$$

$\ln 10 = 2.3026$ $\quad 2 \ln 10 = 4.6052$ $\quad 3 \ln 10 = 6.9078$

$4 \ln 10 = 9.2103$ $\quad 5 \ln 10 = 11.5129$ $\quad 6 \ln 10 = 13.8155$

ANSWERS TO ODD-NUMBERED PROBLEMS

(Some answers have intentionally been omitted.)

Exercises 2, Page 9

1. $<$ **3.** $<$ **5.** $>$ **7.** $<$
9. 16 **11.** 27 **13.** -5 **15.** -27
17. -4

Exercises 3, Pages 11–12

1. 34 **3.** -72 **5.** -102 **7.** $-\frac{1}{2}$ **9.** -16

11. (a) $c = 56$ cents
(b) $c = 8n$ cents
(c) $c = kn$ cents

13. (a) $d = 275$ mi
(b) $d = 5v$ mi
(c) $d = tv$ mi

15. (a) $V = 135$ in.^{3}
(b) $V = abc$ in.^{3}

17. (a) $A = \$9.75$
(b) $A = (0.25x + 0.50y + z)$ dollars

19. (a) $p = 11$ cents
(b) $p = \dfrac{ax + by + cz}{x + y + z}$ cents

21. (a) $f = \frac{9}{20}$
(b) $f = \dfrac{t}{x} + \dfrac{t}{y}$

23. (a) $w = 6.4$ oz
(b) $w = 0.01xy$ oz

Exercises 4, Pages 14–15

1. (a) $7x - 2y + 2$
(b) $3x - 4y + 10$

3. (a) $-9x - y - 12z$
(b) $-5x + 5y + 6z$

5. (a) $-2u - 4v - w + 1$
(b) $8u - 6v - 3w + 13$

7. $x - y + 13z + 9$

9. $9x + 4y$ **11.** $-10r - 2s$

13. $2d - 6e - 12f$ **15.** $6 - (2x - y + 3z)$

17. $18 - (-u - 4v - 7w)$ **19.** $-a - (4b - 3c + 6d)$

Exercises 5, Page 17

1. 25 **3.** -64 **5.** $\frac{1}{8}$ **7.** $\frac{16}{81}$
9. $-\frac{1}{32}$ **11.** -243 **13.** -3000 **15.** -96
17. y^9 **19.** u^{m+3} **21.** b^6 **23.** x^6y^6
25. $-27r^9s^{15}$ **27.** $32x^{10}$ **29.** $-21x^3y^8$ **31.** $18d^5e^8$
33. $-128a^4b^8c^9$ **35.** -15 **37.** -10 **39.** $\frac{35}{72}$
41. Undefined **43.** 11

Exercises 6, Pages 22–23

1. $x^3 + x^2 - 19x + 21$ **3.** $z^3 + z^2 - 14z - 24$

5. $2a^4 - a^3 - 14a^2 + 19a - 6$ **7.** $4x^2 + y^2 + 9z^2 + 4xy - 12xz - 6yz$

9. $2x^4 + 7x^3y - 17x^2y^2 + 2xy^3 + 6y^4$

11. $c^2 + 4d^2 + e^2 + f^2 - 4cd + 2ce - 2cf - 4de + 4df - 2ef$

13. $\dfrac{5y^3}{18x^3}$ **15.** $-\dfrac{2d^2}{5e^2f^7}$ **17.** $\dfrac{4x^4z^{15}}{3y^8}$ **19.** $\dfrac{-125x^3}{8y^6z^{15}}$

21. $\dfrac{4x}{y} - \dfrac{2y}{x}$ **23.** $\dfrac{c^2}{d^2} + 4 + \dfrac{d^2}{c^2}$ **25.** $\dfrac{z}{2} - \dfrac{3}{2} + \dfrac{7}{2z} - \dfrac{2}{z^3}$

27. $x^2 - 2x - 3$ **29.** $8a^3 + 4a^2b + 2ab^2 + b^3$

31. $y^2 - 2y + 1 + \dfrac{6}{2y + 1}$ **33.** $m + 2$

35. $3k^2 - k + 5$ **37.** $x^2 + xy + y^2 - 2xz + 4z^2 + 2yz$

Exercises 7, Pages 23–24

1. (*a*) $2x^3 + 6x^2 + 10x + 7$
(*b*) $6x^3 - 12x^2 - 12x + 7$

3. (*a*) $2x + 17y + 10z - 12$
(*b*) $8x - 21y - 4z + 12$

5. (*a*) $14ab - bc - 56ca$
(*b*) $32ab + 21bc - 14ca$

7. $x^9 - 8y^9$

9. $1 + 125k^6$ **11.** $32s^4 + 12s^3 - 66s^2 + 3s + 14$

13. $24u^4 + 2u^3w - 35u^2w^2 + 4uw^3 + 5w^4$

15. $-20c$ **17.** $1 - 2b$

19. $y^2 + 2y + 2$ **21.** $5a^2 + 2a - 1$

23. $x^2 - x + 1$ **25.** $3a^2b^4 + 5ab^2 - 6$

27. $1 + \dfrac{4}{y - 1}$ **29.** $3v + 6 + \dfrac{4}{v - 2}$

31. $2c + \dfrac{13}{2} + \dfrac{17/2}{2c - 3}$ **33.** $(8x + 11y + 6z)$ cents

35. $\dfrac{m}{a} + \dfrac{n}{b}$ **37.** $n(r - h)$mi

Exercises 8, Page 29

7. 4 **9.** -2.8 **11.** 33 **13.** 16

15. 5 **17.** 1 **19.** 4 **21.** $\frac{5}{6}$

23. 3.4 **25.** 3 **27.** $-\frac{5}{26}$ **29.** 1

Exercises 9, Pages 33–34

1. 96 ft^2 **3.** 2300π ft^2, 10.6π ft

5. 338.24 ft^2, 412.38 ft^3 **7.** 324π ft^3, 108π ft^2

9. 1024π cm^3, 320π cm^2 **11.** 972π in.^{3}, 324π in.^{2}

13. $\dfrac{V}{\pi r^2}$ **15.** $\dfrac{2s}{t^2}$

17. $\dfrac{A - \pi r^2}{\pi r}$ **19.** $l - dn + d, \dfrac{l - a + d}{d}, \dfrac{l - a}{n - 1}$

21. $\dfrac{km_1m_2}{F}$ **23.** $\dfrac{P_2V_2}{P_1}, \dfrac{P_2V_2}{V_1}$

25. $s - rs, \dfrac{s-a}{s}$

27. $\dfrac{fv}{v-f}, \dfrac{fu}{u-f}, \dfrac{uv}{u+v}$

Exercises 10, Pages 37–39

1. 57, 133
3. 13 in.
5. 8 in. by 10 in.
7. $\dfrac{1}{2\pi}$ ft
9. 847
11. A: \$30, B: \$15, C: \$34
13. \$60, \$108
15. \$18,000 at $2\frac{1}{2}\%$
    \$7,000 at 3%
17. A: 96 lb, B: 112 lb
19. 45 lb at 70¢,
    55 lb at 90¢
21. 16 lb of 35%,
    4 lb of 65%
23. 25.2 parts approx.
25. 375 mph, 450 mph
27. 17 yd
29. $7{:}05\frac{5}{11}$ o'clock
31. 4 days

Exercises 11, Pages 42–43

1. $v^2 + 14v + 48$
3. $6x^2 + x - 35$
5. $12a^2b^2 + 7ab - 10$
7. $64c^4 - 49$
9. $9a^2 - 30a + 25$
11. $m^6 - 15m^4 + 75m^2 - 125$
13. $64 - 9b^2c^2$
15. $81u^4 - 36u^2v^4 + 4v^8$
17. $6x^2 - 50xh + 56h^2$
19. $a^2 + 8ab + 16b^2 - 25$
21. $x^2 + 4y^2 + 4 - 4xy - 4x + 8y$
23. $16w^4 - 52w^2k + 22k^2$
25. $r^2 + 4s^2 + t^2 + 1 - 4rs + 2rt + 2r - 4st - 4s + 2t$
27. $2a^2 - a(b + c) - 3(b + c)^2$
29. 2016
31. 7209
33. 7225
35. 18,225

Exercises 12, Pages 44–45

1. $5y^3(x^2 + 6y^2)$
3. $2c(7d + 3e + f)$
5. $5ab^2(3 - 4ab - 5a^4b^5)$
7. $(5x + 3y^2)(5x - 3y^2)$
9. $(4cd^3 + 9e^4)(4cd^3 - 9e^4)$
11. $(x + 6y)(x - 4)$
13. $(4c^2 + m)(6 - r^2)$
15. $2k^3(2hk + 3)(2hk - 3)$
17. $(x^2 + 4y^2)(x + 2y)(x - 2y)$
19. $(r + 2s + 5)(r + 2s - 5)$
21. $(x + y + 1)(x + y - 1)$
23. $(b - 2)(x^2 - 3x + 1)$
25. $(5x - 4y)(s + 3h - k)$
27. $5(x^3 - 3y)(7x - 3y)$
29. $(s - t + 5c - d)(s - t - 5c + d)$
31. $7x^6(2x^2 - 3)^5(5x^2 - 3)$
33. 1280
35. 4800
37. -1431

Exercises 13, Page 47

1. $(x + 6)(x + 2)$
3. $(4 - t)(6 + t)$
5. $(3w + 2)(w - 1)$
7. $(3 - y)^2$

9. $(v - 6)(v - 3)$
11. $(3 - x)(1 + 2x)$
13. $(z + 3)^2(z - 3)^2$
15. $(3x - 4y)^2$
17. $3(y + 4a)(y - 2a)$
19. $(5v - 2)(4v - 3)$
21. $(6m - a)(m - 12a)$
23. $(8bx + 3a)(3bx - 4a)$
25. $(a - b + 5)(a - b - 2)$
27. $(c - 2m + 2)(c - 2m - 1)$

Exercises 14, Page 48

1. $(y + 3)(y^2 - 3y + 9)$
3. $(b^4 - 5c^5)(b^8 + 5b^4c^5 + 25c^{10})$
5. $(z^2 + 2z + 2)(z^2 - 2z + 2)$
7. $(d^4 + xy)(d^8 - d^4xy + x^2y^2)$
9. $(a^2 - 3a + 3)(a^2 + 3a + 3)$
11. $(2x - y)(4x^2 + 2xy + y^2 - 1)$
13. $(2v + 1)(2v - 1)(v + 2)(v - 2)$
15. $(x - 3y^2)(x^2 + 3xy^2 + 9y^4 + x + 3y^2)$
17. $[(c - d) - (a - 2b)][(c - d)^2 + (c - d)(a - 2b) + (a - 2b)^2]$
19. $[2(a + b)^2 + 2(a + b) + 1][2(a + b)^2 - 2(a + b) + 1]$

Exercises 15, Pages 49–50

1. $3(5 + y)(5 - y)$
3. $2(b + 3)(b^2 - 3b + 9)$
5. $r^4(r - 4)(r + 3)$
7. $(7 + x)(3 - x)$
9. $(5m - 4)^2$
11. $(7 + 6ab^2)(7 - 6ab^2)$
13. $(4v - 7)(3v + 4)$
15. $(d^4 - c^5)(d^8 + d^4c^5 + c^{10})$
17. $(a + 3b)(a - 3b - 1)$
19. $(8 + c + 3d)(8 - c - 3d)$
21. $(e^2 + e + 4)(e^2 - e + 4)$
23. $(a^2 + 6)(a - 5)$
25. $(2x + 3)(2x - 3)(3x^2 + 5)$
27. $(ab + c^2)(a^2b^2 - abc^2 + c^4)(ab - c^2)(a^2b^2 + abc^2 + c^4)$
29. $(m + 2)(m - 2)^2$
31. $(b - 4)^3$
33. $(4m + p + y)(4m + p - y)$
35. $(2 + 5r)(2 - 5r + u)$
37. $(1 - h)(1 + h + h^2)(4 + 3s)(4 - 3s)$
39. $(x + 2)^4$
41. $y(5y + 1)(3y - 5)(y + 2)$
43. $(x + 1)(2x^2 + 3x - 3)$
45. $(v + 4)(v - 3)(v + 3)(v - 2)$
47. $(k + 4)(2h - y - 3)$
49. $(s + t + r - 3)(s + t - r + 3)$

Exercises 16, Pages 54–55

1. $y = 3$
3. $e = 4$ or $e = -3$
5. $v = 0$ or $v = 6w$
7. $\dfrac{4b^2c}{7}$
9. $\dfrac{8r^5s^5}{5t^4}$
11. $\dfrac{3u^3v}{4v - 6u}$
13. c^3d^3
15. $\dfrac{13u^2w^3z^2}{22vxy^3}$
17. $\dfrac{3t^2}{4}$
19. $\dfrac{2v + 3}{3v - 5}$
21. $-\dfrac{4b + 5}{5b + 6}$
23. $\dfrac{(x + 3)(x - 4)}{(2x + 1)(x - 1)}$
25. $\dfrac{z}{z - 3}$
27. $\dfrac{2eh^2}{3k^3(3e + 7)}$
29. -1
31. 1

Exercises 17, Pages 59–60

1. $\dfrac{9c + 7a + 12b}{252abc}$
3. $\dfrac{105y^4 + 125x^2y - 4x}{900x^3y^6}$
5. $\dfrac{2m^2 - 25m + 75}{40m(m - 3)}$
7. $\dfrac{1}{1 - x}$
9. $\dfrac{a^6}{a^2 + b^2}$
11. $\dfrac{4x - 9}{(2x - 5)(x - 2)}$
13. $\dfrac{b + 1}{2(b + 4)}$
15. $-\dfrac{7c + 34}{c(c + 5)}$
17. $\dfrac{1}{2a - 1}$
19. $-\dfrac{3a + 6c}{20ac}$
21. $\dfrac{x - 7}{(2x - 7)(3x + 2)}$
23. $\dfrac{3y + 5}{12(y^2 + y + 1)}$
25. $-\dfrac{(a + b)^2}{b^2(b + 1)(b - 1)}$
27. 0

Exercises 18, Pages 62–64

1. $\dfrac{3b^3}{7a^2}$
3. $\dfrac{8x^2y}{3}$
5. $\dfrac{2v + 3}{3v + 1}$
7. $\dfrac{w + 4}{w + 1}$
9. $\dfrac{m + 5}{m - 4}$
11. $\dfrac{a^2 + 2b^2}{ab}$
13. $\dfrac{v^2 + 1}{2v}$
15. $\dfrac{y^3}{9(y - 3)}$
17. $t + r - s$
19. $\dfrac{4x + 6y}{5xy}$
21. $-\dfrac{1 + 2e}{2}$
23. $\dfrac{7y - 4}{2 - y}$
25. $\dfrac{1}{x}$
27. $1 - x$

Exercises 19, Pages 67–68

1. 7
3. 1
5. 7
7. $\frac{10}{3}$
9. -8
11. 2
13. -9
15. $\frac{7}{11}$
17. -6
19. $\dfrac{3}{2}$
21. $-\dfrac{6}{13}$
23. $\dfrac{abr}{as + bs}$
25. $-\dfrac{de}{f^2}$
27. $ra_n + s - rs$, $\dfrac{a_1 + rs - s}{r}$
29. $\dfrac{Z_2Z_t}{Z_2 - Z_t}$

Exercises 20, Pages 70–71

1. 65, 72
3. 27, 90
5. 10 lb
7. 9 at \$18; 36 at \$24
9. 100 mi
11. \$900
13. 97
15. A: 6 days; B: 12 days; C: 6 days
17. 11, 22, and 33 min

19. 12 min
21. 58 days
23. gold: $31\frac{2}{3}$ oz
silver: $13\frac{1}{3}$ oz
25. 42% approx.
27. $17\frac{1}{2}$ hr, $24\frac{1}{2}$ hr

Exercises 21, Pages 74–77

1. 1.63 in. approx.
5. 240 kw approx.
7. 5.3 ft approx.; 130 hp approx.

Exercises 22, Page 79

3. 15 (linear units)^2
5. $(-1, 3)$; $(1, -3)$; $(9, 3)$
7. 10 linear units
9. No
11. 0; 0

Exercises 23, Pages 84–85

9. 0, 6
11. -4, 1
13. -2.5, 3.5
15. -3, 0, 3
17. -4, -2, 2
19. 0.6, 3.4
21. 1.6
23. -8, 1
25. -3.4, -0.7, 0.6
27. 6 and 6
29. 7 in. by 7 in.
31. 20,000 yd^2
33. 2 in. by 2 in.
35. 300 units per week

Exercises 24, Pages 88–89

1. $A = \dfrac{C^2}{4\pi}$
3. $V = \pi h(h + 5)^2$
5. $d^2 = 3e^2$
7. $A = \dfrac{b_1^2}{3}$
9. -28; 42; $-\frac{98}{27}$
11. $-\frac{3}{7}$; 0; -19; $-\frac{225}{17}$
13. $k^2 + 3ak + 2a^2$; 0; $2a^2$; $\dfrac{a^2(1 - 3c + 2c^2)}{c^2}$
15. $\dfrac{y + 35}{5y - 40}$; $\dfrac{z + 7x}{5z - 8x}$; $\dfrac{2 - 7b}{10 + 8b}$
17. $\dfrac{2u + h}{2u^2(u + h)^2}$
19. $\dfrac{x}{9 - 2x}$
21. $u^4 + 2u^2 + 4$
23. $\dfrac{-2bcx^2}{(2c + bx)(2c - bx)}$
25. $h(x, y) = h(-x, y) = h(x, -y) = h(-x, -y)$
29. $6u^2 + 6v^2$

Exercises 25, Pages 92–93

1. (3, 2)
3. $(-2, 5)$
5. (5, 0)
7. No solution
9. (1.5, 0.25)
11. (3, 6)
13. $(-0.75, -1.25)$

Exercises 26, Pages 97–98

1. $(7, -3)$	**3.** $(6, -8)$	**5.** $(3, 5)$
7. $(\frac{3}{4}, \frac{7}{4})$	**9.** $(10, 4)$	**11.** $(4, -2)$
13. $(-1, 5)$	**15.** $(8, 20)$	**17.** $(c - 2d, c + 3d)$
19. $(-a, -b)$	**21.** $\left(\dfrac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}, \dfrac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}\right)$	
23. $(2, 4)$	**25.** $(\frac{1}{3}, \frac{1}{2})$	**27.** $(5, 1)$

Exercises 27, Page 100

(Each solution is given in the alphabetical order of the unknowns.)

1. $(3, -1, -2)$	**3.** $(9, 12, 8)$	**5.** $(\frac{1}{4}, \frac{2}{3}, \frac{1}{6})$
7. $\left(\dfrac{c - b - a}{2c}, \dfrac{a + b + c}{2b}, \dfrac{a - b + c}{2a}\right)$		**9.** $(\frac{1}{2}, 1, -\frac{1}{3})$
11. $(-2, 2, 1, -3)$	**13.** $(2, 1, -2, 3)$	**15.** $(\frac{1}{6}, \frac{1}{3}, \frac{1}{2}, -\frac{1}{4})$

Exercises 28, Pages 102–105

1. $\frac{3}{16}$

3. Longs Peak: 14,255 ft
Pikes Peak: 14,110 ft

5. \$18,000 at 2%;
\$12,000 at $2\frac{1}{2}$%

7. skis: \$22
bindings: \$3.50

9. $a = \frac{9}{5}$, $b = 32$

11. 14 ft

13. 360 mph, 60 mph

15. 30 mph, 450 mph

17. 30 mi, 40 min

19. $1\frac{1}{2}$ gal of water;
$4\frac{1}{2}$ gal of 40% solution

21. \$10.40; 16 quarters, 64 dimes

23. A: 40 days; B: 120 days;
C: 60 days

25. larger pipe: 90 hr
smaller pipe: 180 hr

27. $a = -\frac{3}{8}$, $b = \frac{11}{4}$, $c = -5$

Exercises 29, Pages 109–110

3. Irrational: $\sqrt{3}$, $\sqrt[3]{12}$, π, $2 + \sqrt{5}$
Rational: 1.7, 3.27, 0.33, $\sqrt[4]{16}$, $\sqrt[3]{64}$, 0.0011

5. $2\sqrt{3}$; $\dfrac{7\sqrt{3}}{2}$; 9; $\dfrac{\sqrt{15}}{2}$; no

Exercises 30, Page 112

1. 1.304	**3.** 4.359	**5.** 22.136	**7.** 36.056
9. 0.933	**11.** 0.251	**13.** 0.057	**15.** 1.613
17. 3.756	**19.** 0.772	**21.** 5.540	**23.** 13.006
25. 39.791	**27.** 0.262	**29.** 76.631	**31.** 0.134
33. 5.650	**35.** 10.708	**37.** 0.472	**39.** 1.390

Exercises 31, Pages 116–117

1. $5^{1/2}$
3. $11^{1/4}$
5. $(7y^5)^{1/4}$
7. $(-56u^5)^{1/3}$
9. $\sqrt[4]{5^3}$
11. $\sqrt{(6x)^3}$
13. $\sqrt[5]{(-b)^2}$
15. $3\sqrt[5]{-7x}$
17. $6\sqrt[3]{ab^2}$
19. 16
21. 625
23. 4
25. $-\frac{1}{32}$
27. 0.000216
29. 0.00000081
31. $15x^{7/12}y^{7/6}$
33. $14uv$
35. $e^{23}f^{17}$
37. r^7s^{10}
39. $w^{m^2+k^2}z^{2km}$
41. $a^{4/3} + 2a^{2/3}b^{2/3} + b^{4/3}$
43. $27x^2 - 27x^{4/3}y^{2/3} + 9x^{2/3}y^{4/3} - y^2$
45. $8a^5 - b^5$
47. $x + y$
49. $\dfrac{2a^{3/2}b^{3/2}}{3}$
51. $\dfrac{11c^{1/2}}{6a^{1/2}b^{1/2}}$
53. y^6
55. $\dfrac{d^{9s}}{e^{2r}}$
57. $3x^{1/2} - 2y^{1/2}$
59. $9a^{4/3} + 6a^{2/3}b^{2/3} + 4b^{4/3}$
61. 125
63. 625
65. -8

Exercises 32, Pages 120–122

1. $\dfrac{1}{u^3v^6}$
3. $\dfrac{4b^2}{a^7}$
5. $2x^5y^4$
7. $\dfrac{(u - v)^2}{4}$
9. $\dfrac{25r^{14}t^{10}}{s^4}$
11. $\dfrac{a^3}{b}$
13. $\dfrac{d^{18}}{c^{26}}$
15. $-\dfrac{u^3}{3v^2w^4}$
17. $-\dfrac{6a}{b^2}$
19. $a^{-8} - b^{-8}$
21. $c^{-3} - y^{-6}$
23. $\dfrac{y^4 - 2y^2x^2 + 4x^4}{x^4y^4}$
25. $6h^5 + 7h$
27. $y^{1/2}$
29. b
31. $\dfrac{1 - v}{2v - 1}$
33. xy^2
35. $\dfrac{4y^{1/2}}{y - 4}$
37. $\dfrac{e^{2y} + 1}{e^y}$
39. $\frac{1}{9}$
41. $-\frac{1}{8}$
43. $-\frac{1}{32}$
45. 1
47. 5
49. b^y
51. $\dfrac{1}{y^b}$
53. 0
55. $\frac{1}{2}$
57. $3v$
59. 9
61. 8
63. 9

Exercises 33, Pages 126–127

1. $7\sqrt{2}$
3. $3\sqrt[3]{3}$
5. $3\sqrt[4]{2}$
7. $-2\sqrt[5]{3}$
9. $\dfrac{\sqrt{2}}{20}$
11. $6a^2b^6\sqrt{10b}$

13. $-5a^2b^3\sqrt[3]{5a^2b}$
15. $3ab^3\sqrt[3]{a^2}$
17. $x^3y^5\sqrt[k]{y^2}$
19. $|x-3|\sqrt{3}$
21. $|y^2-5y|\sqrt{3y}$
23. $\sqrt{54}$
25. $\sqrt[3]{189}$
27. $\sqrt[3]{6ab^3}$
29. $\sqrt[4]{\dfrac{2s^3}{rt}}$
31. $\sqrt[m]{x^2y^3}$
33. $\sqrt{\dfrac{u-2w}{u+2w}}$
35. $\frac{2}{3}\sqrt{6}$
37. $\frac{1}{5}\sqrt[3]{55}$
39. $\dfrac{x^2y}{4z^4}\sqrt{6xyz}$
41. $\dfrac{-s\sqrt[3]{42rs^2}}{7r}$
43. $\dfrac{-a\sqrt[5]{4a^3b^4c^4}}{2b^5c^4}$
45. $\dfrac{e^3f^4\sqrt[3]{e+2f}}{e+2f}$
47. $d^2\sqrt{6cd}$
49. $r\sqrt[3]{25r^2s^2}$
51. $c^2d^3\sqrt{cd}$
53. $\sqrt[6]{3125}$

Exercises 34, Page 129

1. $5\sqrt{3}$
3. $-7\sqrt{5}$
5. $13\sqrt[3]{3}$
7. $10d^2f\sqrt{f}$
9. $19z^3\sqrt{5z}$
11. $\dfrac{-5\sqrt{2}}{12}$
13. $\frac{1}{3}\sqrt[3]{12}$
15. $\dfrac{r}{3s^2}\sqrt{3rs}$
17. $2b\sqrt[3]{5b^2}$
19. $7v^2\sqrt[3]{2v}+6v^2\sqrt{3v}$
21. 0
23. $4\sqrt{e^2-f^2}$

Exercises 35, Pages 131–132

1. 30
3. 3
5. $\frac{1}{3}$
7. $2\sqrt[6]{2}$
9. $14r^5s^3t^3$
11. $24\sqrt{5}-40\sqrt{2}+40$
13. $17-4\sqrt{15}$
15. $40+2x-8\sqrt{24+3x}$
17. -13
19. $18+21\sqrt{2}$
21. $9+2\sqrt{35}$
23. 0
25. $-\dfrac{19+7\sqrt{3}}{2}$
27. $\dfrac{3\sqrt{5}}{20}$
29. $\sqrt[6]{243}$
31. $\dfrac{x^3}{7y^4}\sqrt{7y}$
33. $\dfrac{\sqrt[4]{27}}{3}$
35. $\sqrt[6]{56}-\sqrt[3]{3}$
37. $-\dfrac{5\sqrt{2}+4\sqrt{5}}{3}$
39. $\dfrac{5+2\sqrt{42}}{13}$
41. $\dfrac{4\sqrt{5}-5\sqrt{6}+3\sqrt{30}-12}{7}$
43. $-\dfrac{2c^2+d^2+2c\sqrt{d^2+c^2}}{d^2}$
45. $\dfrac{\sqrt{r+4}-\sqrt{r}}{2}$
47. $\dfrac{1}{6-\sqrt{35}}$
49. $\dfrac{1}{\sqrt{x+y}+\sqrt{x}}$
51. 7.162
53. 0.924
55. 3.098

Exercises 36, Page 134

1. 9 3. 8 5. No root 7. 6
9. $-\frac{4}{3}$ 11. $\frac{95}{9}$ 13. $7c$ 15. 12
17. $4c$

Exercises 37, Pages 135–136

1. $C = 2\sqrt{\pi A}$
3. $A = \dfrac{3\sqrt{3}}{2}s^2$
5. $A = \dfrac{\sqrt{3}}{3}h^2$
7. $h = a\sqrt{7},\ V = \dfrac{4\sqrt{7}}{3}a^3$
9. $24\sqrt{3}$ in.^{2}
11. $49, 16$
13. $h = \dfrac{\sqrt{6}}{3}e,\ V = \dfrac{\sqrt{2}}{12}e^3$
15. $\dfrac{14(\sqrt{2}+1)}{5}$ ft

Exercises 38, Pages 138–139

1. $17 + i$ 3. $-18 + i6\sqrt{2}$ 5. $5\sqrt{3} + i9\sqrt{5}$
7. $-3 - 22i$ 9. $1 + 41i$ 11. $28 + i\sqrt{15}$
13. i 15. $3 + 14i$ 17. $22i$
19. $1 + i$ 21. $8 + i$ 23. $-4 + i5\sqrt{3}$

Exercises 39, Page 139

1. $\dfrac{8b^3c^{21}}{a^4}$ 3. 1 5. 8192
7. 32 9. $\sqrt{7} - \sqrt{3}$ 11. $\sqrt{6} + \sqrt{2}$
13. $\dfrac{\sqrt{3} - \sqrt{7}}{2}$ 15. $2(3 + \sqrt{3})\sqrt{3 - \sqrt{3}}$ 17. $\dfrac{4\sqrt[3]{a^2} - 2\sqrt[3]{ab} + \sqrt[3]{b^2}}{8a + b}$
19. $\frac{5}{2}$ 21. $\left|\dfrac{n^3}{r^9}\right| z^{9/4}$ 23. 12

Exercises 40, Page 141

1. $7y^2 + 2y + 11 = 0$
3. $6v^2 + 4v + 39 = 0$
5. $z^2 + 8kz + (5k - 3) = 0$
7. $6u^2 - 23u + 30 = 0$
9. $4x^2 + 7x - 17 = 0$
11. $\pm\sqrt{3}$
13. $\dfrac{\pm 2\sqrt{3a}}{3}i$ 15. $\pm\sqrt{6}$ 17. 13

Exercises 41, Page 143

1. $0, \frac{8}{3}$ 3. $4, -3$ 5. $-5, \frac{4}{3}$ 7. $\frac{5}{2}, \frac{5}{2}$

9. $-2b,\ 2b+1$ **11.** $2+3cd,\ 2-3cd$ **13.** $-4a,\ 2a$

15. $\dfrac{-3h^2}{7},\ 2h^2$ **17.** $0,\ \pm 8i$ **19.** $-2b,\ 2b,\ 2b$

21. $\dfrac{c-2d}{3},\ \dfrac{2d-c}{3}$ **23.** $2d,\ 2d+1$

Exercises 42, Pages 144–145

1. $6,\ -4$ **3.** $3\pm\sqrt{2}$ **5.** $-3,\ \frac{2}{3}$ **7.** $-\frac{3}{7},\ 2$

9. $\frac{1}{4},\ \frac{5}{4}$ **11.** $-5\pm\sqrt{6}$ **13.** $1,\ \frac{7}{3}$ **15.** $\dfrac{2\pm i\sqrt{11}}{3}$

Exercises 43, Pages 146–147

1. $-10,\ 6$ **3.** $\frac{3}{5},\ 2$ **5.** $-1\pm\sqrt{3}$

7. $-\frac{3}{4},\ \frac{2}{7}$ **9.** $3\pm\sqrt{5}$ **11.** $\dfrac{2\pm 7i}{3}$

13. $\dfrac{-4\pm i\sqrt{3}}{5}$ **15.** $\dfrac{3\pm i\sqrt{47}}{14}$ **17.** $\dfrac{4\pm\sqrt{11}}{2}$

19. $-\sqrt{3}\pm 2$ **21.** $\dfrac{-2\sqrt{5}\pm i\sqrt{5}}{5}$ **23.** $\dfrac{-b\pm\sqrt{b^2-ac}}{a}$

25. $\dfrac{1\pm\sqrt{1-8n}}{4y}$ **27.** $3\pm\sqrt{9-5v-v^2}$ **29.** $\dfrac{-v_0\pm\sqrt{v_0^2+2gs}}{g}$

31. $\dfrac{-3y\pm\sqrt{13y^2+16}}{2};\ \dfrac{3x\pm\sqrt{13x^2-16}}{2}$

Exercises 44, Page 148

1. $(2x+7)(x-4)$ **3.** $(5z-4)(2z-3)$

5. $(4+7w)(2-w)$

7. $5\left(z-\dfrac{3+\sqrt{29}}{10}\right)\left(z-\dfrac{3-\sqrt{29}}{10}\right)$

9. $7\left(y-\dfrac{1+i\sqrt{55}}{14}\right)\left(y-\dfrac{1-i\sqrt{55}}{14}\right)$

11. $(4r+1-\sqrt{6})(4r+1+\sqrt{6})$ **13.** $(3x+4y)(2x-3y)$

15. $(5bc-d)(2bc-3d)$ **17.** $[3x+b-1][(b+3)x-2]$

Exercises 45, Page 152

1. $(6,\ -9)$ **3.** $(-2,\ 2)$ **5.** $(2,\ 16)$

7. $(2,\ 4)$ **9.** 9 and 9 **11.** 31,250 yd^2

13. -4 and 4 **15.** 5 in. **17.** 45 yd by 135 yd

Exercises 46, Page 154

1. Real, unequal, irrational
3. Real, equal, rational
5. Imaginary
7. Real, unequal, rational
9. Imaginary

Exercises 47, Page 157

1. 16; 40
3. $\frac{7}{5}$; $-\frac{8}{5}$
5. -6; $-\frac{10}{3}$
7. $ab - a^2 - b^2$; $\dfrac{6ab}{a + b}$
9. $3h$; $-16h$
11. $x^2 - 2x - 35 = 0$
13. $x^2 + 13x + 36 = 0$
15. $x^2 + 4x - 14 = 0$
17. $x^2 - 10x + 89 = 0$
19. $x^2 + 6x + 29 = 0$

Exercises 48, Pages 158–159

1. $\frac{8}{5}$
3. $-7, 2$
5. $\frac{25}{7}$
7. $-1, \frac{7}{2}$
9. $k = -\frac{19}{2}$, $m = -\frac{9}{2}$
11. $k = -\frac{5}{2}$, $m = 2$
13. 10
15. -20
17. 5
19. 14
21. 12
23. 42
25. 4
27. 2
29. 4

Exercises 49, Pages 161–162

1. $-4, 2 \pm i2\sqrt{3}$
3. $-\frac{1}{5}, \dfrac{1 \pm i\sqrt{3}}{10}$
5. $\pm\frac{1}{2}, \dfrac{1 \pm i\sqrt{3}}{4}, \dfrac{-1 \pm i\sqrt{3}}{4}$
7. $-\frac{3}{5}, \frac{7}{4}$
9. $-1, 7$
11. 2, 8
13. $-2, 4$
15. 9
17. 8
19. $-2, \frac{1}{4}$
21. $a, b \quad (a + b \neq 0)$
23. $\dfrac{c}{c - d}, \dfrac{d}{c - d}$
25. $\dfrac{2k}{h}, -\dfrac{h}{2k}$

Exercises 50, Pages 163–164

1. $\pm 2, \pm 3$
3. $\pm\frac{1}{2}, \pm\dfrac{\sqrt{5}}{5}$
5. $\pm\frac{1}{2}, \pm 1, \pm\dfrac{i}{2}, \pm i$
7. $-27, 64$
9. -3
11. $-4, 1$
13. $-\frac{7}{2}, 3$
15. $-5, -1, -1, 3$
17. $-1, 1, 2, 4$
19. $-2, -1, \frac{1}{3}, \frac{2}{3}$
21. $11 + 4\sqrt{7}, 1, 9$

Exercises 51, Pages 167–168

1. $\frac{3}{7}$ or $\frac{7}{3}$
3. Base: 28 in. altitude: 18 in.
5. $(4 + 4\sqrt{2})$ in.
7. $(8 - 4\sqrt{2})$ ft
9. 12 in. by 24 in.
11. 6 ft

13. 5 in., 2 in. **15.** $2\frac{1}{2}$ in. or $4\frac{1}{2}$ in. **17.** 17.2 min, 6 min
19. 480 mph **21.** 32 **23.** 36 min, 45 min
25. 25 days **27.** 493 ft approx. **29.** 50 ft, 58 ft

Exercises 52, Page 171

1. $\frac{3}{16}$ **3.** $\frac{325}{2}$ **5.** $\frac{2}{9}$
7. 106; 212; 318 **9.** 128; 256; 320; 576 **11.** $\frac{11}{16}$
13. $-\frac{5}{3}, \frac{7}{9}$

Exercises 53, Pages 176–177

7. (*a*) $\pm 5\sqrt{15}$; (*b*) ± 42; (*c*) $\pm(e-f)\sqrt{e^2+ef+f^2}$; (*d*) $\pm 10(c+d)\sqrt{c-d}$
9. 12 and 20 units, respectively **11.** $4\frac{2}{7}$ ft
15. (*a*) 5 : 8; (*b*) 9 : 11 **19.** (*a*) 1 : 2; (*b*) 5 : 6

Exercises 54, Pages 182–184

1. 405 **3.** 2916 **7.** $\frac{1}{16}$
9. 11.4 times approx. **11.** 64.8 lb **13.** 25
15. 40 dynes; 1440 dynes **17.** 42 mi **19.** 16; 1
21. 5250 ft^3 **23.** 5.25 : 1 **25.** $13\frac{1}{3}$ ft-c
27. 300 lb **29.** 5.23 ohms approx. **31.** 3.75 ohms
33. $\dfrac{v^2 + 25{,}000}{500v}$ dollars/mi

Exercises 55, Page 187

1. 868. **3.** 8.04 **5.** 260.1 **7.** 1.653
9. 47.9 **11.** 242.6 **13.** 23.12 **15.** 4.197

Exercises 56, Pages 190–191

1. 6×10^5 **3.** 5.671×10^7 **5.** 7.293×10^3
7. 6.4×10^4 **9.** 2.43×10^{-1} **11.** 5.92×10^{-2}
13. 3.648×10^{-2} **15.** 4.08×10^{-2} **17.** 1.60×10^4
19. 2.8×10^2 **21.** 2.28 **23.** 2.2×10
25. 8.67×10^{-1} **27.** 1.3×10^3 **29.** 3.0×10^4
31. 3.0×10^{-1} **33.** 4.00×10 **35.** 1.76
37. 1.90×10^2 **39.** 372 mm^2 **41.** 64.8 in.^{2}
43. 51.2 in. **45.** 7.194 ft
47. 3.4×10^2 mph; by not more than 18 sec

Exercises 57, Page 194

1. $\log_2 32 = 5$ **3.** $\log_{49} 7 = 0.5$ **5.** $\log_{10} 10{,}000 = 4$

7. $\log_{343} 7 = \frac{1}{3}$
9. $\log_{10} 2 = 0.30103$
11. $\log_a M = 4$
13. $7^2 = 49$
15. $3^{-4} = \frac{1}{81}$
17. $16^{0.75} = 8$
19. $10^{2.30103} = 200$
21. $10^{0.69897} = 5$
23. 3
25. 4
27. 4
29. 5
31. 81
33. 125
35. 27
37. 100,000
39. $\frac{1}{343}$
41. 625
43. 5
45. 3
47. 121
49. 27
51. $\frac{1}{25}$
53. $\frac{1}{256}$

Exercises 58, Pages 198–199

1. 1.38021
3. 1.87506
5. 0.85733
7. −0.26761
9. 3.23754
11. 0.34062
13. 0.28578
15. −0.15918

Exercises 59, Page 201

1. 0
3. 4
5. −1
7. 3
9. −1
11. 1
13. −5
15. 8

Exercises 60, Page 203

1. 9 −10
3. 8 −10
5. 8 −10
7. 8 −10
9. 9 −10
11. 6 −10

Exercises 61, Page 205

1. 1.87204
3. 3.26411
5. 8.64058 −10
7. 9.10823 −10
9. 5.21537
11. 7.93480 −10
13. 8.82847 −10
15. 5.92007
17. 2.760
19. 4471
21. 0.6768
23. 0.002049
25. 0.09553
27. 364.8
29. 0.0008710
31. 10.03

Exercises 62, Page 208

1. 2.67352
3. 3.90538
5. 9.09948 −10
7. 8.56063 −10
9. 2.99491
11. 1.03910
13. 5.79135 −10
15. 6.01271
17. 45.203
19. 0.55502
21. 136.13
23. 0.0070037
25. 0.00089174
27. 0.038705
29. 3347.2
31. 1.6606

Exercises 63, Page 212

1. 71.852
3. −1343.3
5. −11.451
7. 1.1626
9. −0.0045724
11. 3.9437
13. 9.8432
15. 28.383
17. 1.5142×10^{-7}

19. 39.931
21. 0.0023204
23. 23,894
25. 0.36361
27. 0.20339
29. 5.2929
31. 0.77256
33. 1.9053
35. 0.090886
37. 3.5993
39. −0.21922
41. −0.31721
43. 0.99078
45. 166.5
47. 2928
49. 0.55165

Exercises 64, Pages 215–216

1. 29.332
3. 26.627
5. 1.9818
7. 2.7037
9. 6.0749
11. 0.24179
13. 6.3053
15. 75.292
17. 5.21 lb
19. 2.23 lb
21. 597 hp
23. 15.7 ft^3

Exercises 65, Pages 220–221

1. 3.3468
3. 2.7318
5. −2.0612
7. 2.2894
9. 0.9067
11. 4.677
13. 125.8
15. 0.7272
17. 0.2150
19. 2.8074
21. 3.9419
23. 0.1150
25. 2.0794
27. −12.380
29. 1.778
31. 5.06×10^{-3} coulomb
33. 4.87 approx.

Exercises 66, Page 223

1. 1.7712
3. 1.1002
5. 0.48416
7. 0.94674
9. 5.4378
11. 0.29624
13. 1.822
15. 20.80
17. 0.24713
19. 34.33
21. (3.1746, 0.9127)
23. 14
25. $\sqrt[3]{cx^2}$
27. $\frac{1}{2}\ln(y \pm \sqrt{y^2 - 1})$
29. $\dfrac{L}{R}\ln\dfrac{E}{E - IR}$

Exercises 67, Pages 226–227

1. $x^4 + 12x^3 + 54x^2 + 108x + 81$
3. $a^8 - 8a^6b + 24a^4b^2 - 32a^2b^3 + 16b^4$
5. $16d^{12} - 96d^9c^5 + 216d^6c^{10} - 216d^3c^{15} + 81c^{20}$
7. $a^{-8} - 4a^{-6}b^{-2} + 6a^{-4}b^{-4} - 4a^{-2}b^{-6} + b^{-8}$
9. $x^{10} + 5x^8y + 10x^6y^2 + 10x^4y^3 + 5x^2y^4 + y^5$
11. $16h^4 + \frac{32}{3}h^3k + \frac{8}{3}h^2k^2 + \frac{8}{27}hk^3 + \frac{1}{81}k^4$
13. $u^{12} - 12u^{10} + 60u^8 - 160u^6 + 240u^4 - 192u^2 + 64$
15. $\dfrac{a^{12}}{b^4} - \dfrac{4a^6}{b^2} + 6 - \dfrac{4b^2}{a^6} + \dfrac{b^4}{a^{12}}$
17. $128 - 448f^2 + 672f^4 - 560f^6 + 280f^8 - 84f^{10} + 14f^{12} - f^{14}$
19. $\dfrac{e^6}{729} + \dfrac{2\sqrt{5}e^5m}{81} + \dfrac{25e^4m^2}{27} + \dfrac{100\sqrt{5}e^3m^3}{27} + \dfrac{125e^2m^4}{3} + 50\sqrt{5}em^5 + 125m^6$

21. $128a^{14}b^{21} - 224a^{11}b^{18}c + 168a^8b^{15}c^2 - 70a^5b^{12}c^3 + \dfrac{35a^2b^9c^4}{2} - \dfrac{21b^6c^5}{8a} + \dfrac{7b^3c^6}{32a^4} - \dfrac{c^7}{128a^7}$

23. $u^8 - 4u^7 + 10u^6 - 16u^5 + 19u^4 - 16u^3 + 10u^2 - 4u + 1$

25. 1.1262 **27.** 1.3048 **29.** 0.88584

Exercises 68, Page 230

1. $109{,}824y^7$ **3.** $792u^{15}v^{28}$ **5.** $\dfrac{3003x^6z^{16}}{y^6c^8}$ **7.** 41,184

9. $\dfrac{308x^{18}}{243}$ **11.** $\dfrac{1001}{34{,}992y^8}$ **13.** 2,027,520 **15.** 56; 7920; $\frac{1}{36}$

Exercises 69, Pages 232–233

1. $1 - 6x + 24x^2 - 80x^3 + \cdots$

3. $\frac{1}{6}(1 - \frac{5}{6}z + \frac{25}{36}z^2 - \frac{125}{216}z^3 + \cdots)$

5. $u^{-2}\left[1 + \dfrac{1}{2}\left(\dfrac{u}{v}\right)^4 - \dfrac{1}{8}\left(\dfrac{u}{v}\right)^8 + \dfrac{1}{16}\left(\dfrac{u}{v}\right)^{12} - \cdots\right]$

7. $\dfrac{4}{a^2}\left(1 + \dfrac{8}{a^4b^6} + \dfrac{96}{a^8b^{12}} + \dfrac{1280}{a^{12}b^{18}} + \cdots\right)$

9. 1.020 **11.** 0.9849 **13.** 11.09

15. 2.080 **17.** 2.012 **19.** 0.1001

Exercises 71, Pages 243–245

1. (*a*) 5; (*b*) -13; (*d*) $7c^2$; (*f*) $a^2 + 3b^2$; (*g*) $-4e - 5f$

3. $-5\frac{1}{2}$; -37 **5.** $-38v^2 + 95z$; $-68v^2 + 161z$

7. $\dfrac{108a - 126}{5c}$ **9.** 21; 1092 **11.** 100; -68

13. 19; -12 or 2; $13\frac{1}{2}$ **15.** -2; -18 **17.** $k^2 - m^2$

19. 271 **21.** 96 **23.** 19, 31, 43, 55, 67

25. $-8 - 20\sqrt{3}$, $-6 - 16\sqrt{3}$, $-4 - 12\sqrt{3}$, $-2 - 8\sqrt{3}$

27. 45; 14,040 **29.** $5\frac{5}{8}$, $6\frac{3}{7}$, $7\frac{1}{2}$ **31.** \$50.50

33. 21 days **35.** \$71,700 **37.** \$7910

41. 66; 286

Exercises 72, Pages 248–249

1. (*a*) 3; (*b*) $\frac{1}{6}$; (*c*) $-\frac{1}{4}$; (*d*) $-\frac{1}{7}$; (*e*) $\frac{2}{3}$; (*f*) $\frac{2}{5}$; (*g*) $\frac{3}{8}$

3. 486; 4374; 6560 **5.** $\dfrac{a^2}{729b^2}$ **7.** 135; 3

9. -3; -768 **11.** 1024; $819\frac{3}{16}$ **13.** 2; 7

15. $\frac{2}{27}$, $\frac{2}{9}$, $\frac{2}{3}$ **17.** $\pm\frac{3}{5}$ **19.** $2\sqrt{3}$, 6, $6\sqrt{3}$ or $-2\sqrt{3}$, 6, $-6\sqrt{3}$

21. $7\sqrt{2}$, 14, $14\sqrt{2}$, 28, $28\sqrt{2}$ or $-7\sqrt{2}$, 14, $-14\sqrt{2}$, 28, $-28\sqrt{2}$

23. 402,000 approx. **25.** \$2000, \$3125 **27.** $\dfrac{4096}{15625}$; 3.95s approx.

29. $-5, 1, 7$ or $7, 1, -5$ **31.** $\frac{1}{4}$ in.; $\dfrac{\sqrt{3}}{64}$ in.^{2} **33.** 1024π in.^{2}

Exercises 73, Page 252

1. $\frac{7}{9}$ **3.** $\frac{3}{2}$ **5.** $20\frac{5}{6}$
7. $\frac{4}{9}$ **9.** $\frac{67}{99}$ **11.** $\frac{115}{33}$
13. $\frac{392}{55}$ **15.** $\frac{31}{111}$ **17.** 56 ft

Exercises 74, Pages 256–257

(The underlined digits in these answers are uncertain because of the limitations of computation with five-place logarithms.)

1. \$354$\underline{9}$; \$3548.01 **3.** \$12,7$\underline{40}$; \$12,715.98
5. \$17,96$\underline{5}$; \$17,965.17 **7.** \$19,90$\underline{2}$; \$19,901.90
9. \$401.8$\underline{5}$; \$401.85 **11.** \$28,46$\underline{4}$; \$28,465.49 and \$34,$\underline{180}$; \$34,200.31
13. \$16,51$\underline{2}$; \$16,513.23 **15.** \$57,4$\underline{20}$; \$57,427.87
17. 6.6% approx. **19.** \$715.$\underline{60}$; \$715.59

Exercises 75, Pages 262–263

1. 17 lb **3.** 25 lb **5.** 605 mph approx.
7. 10 units **9.** 25 units **11.** $2\sqrt{26}$ units

Exercises 76, Pages 264–265

1. $6 + 8i$; 10 **3.** $12 + 5i$; 13 **5.** $-7 + 7i$; $7\sqrt{2}$ **7.** $2 + 8i$; $2\sqrt{17}$
9. $-6 - 6i$; $6\sqrt{2}$ **11.** $24 + 7i$; 25 **13.** 5; 5 **15.** $9i$; 9

Exercises 77, Page 269

1. 2 **3.** -2 **5.** $-i$
7. $2i$ **9.** $1 + 2i$ **11.** $-169i$; 169
13. $-8 - i8\sqrt{3}$; 16 **15.** $4i$; 4
17. $\dfrac{8 + 6\sqrt{3}}{4} + \dfrac{8\sqrt{3} - 6}{4}i$; 5 **19.** $\dfrac{4\sqrt{7}}{3} + \dfrac{4\sqrt{2}}{3}i$; 4
21. $\frac{8}{5}$; $\frac{8}{5}$ **23.** 1; 1 **25.** $1 + 2i$

Exercises 78, Pages 271–272

1. $x = 3, y = -2$ **3.** $u = \frac{3}{2}, v = -\frac{5}{2}$ **5.** $u = 2, v = 3$
7. $c = -\frac{4}{5}, d = -\frac{4}{5}$ **9.** $x = 3, y = -5$ and/or $u = -1, v = 2$

Exercises 79, Page 275

1. $8(\cos 180° + i \sin 180°)$
3. $18(\cos 90° + i \sin 90°)$
5. $6(\cos 315° + i \sin 315°)$
7. $2\sqrt{5}(\cos 330° + i \sin 330°)$
9. $20(\cos 300° + i \sin 300°)$
11. $2\sqrt{5}(\cos 60° + i \sin 60°)$
13. $5(\cos 323°8' + i \sin 323°8')$
15. $6\sqrt{5}(\cos 116°34' + i \sin 116°34')$
17. $\sqrt{269}(\cos 217°34' + i \sin 217°34')$
19. $10[\cos (45° + k \cdot 360°) + i \sin (45° + k \cdot 360°)]$
21. $23[\cos (180° + k \cdot 360°) + i \sin (180° + k \cdot 360°)]$
23. $16[\cos (150° + k \cdot 360°) + i \sin (150° + k \cdot 360°)]$
25. $8[\cos (210° + k \cdot 360°) + i \sin (210° + k \cdot 360°)]$
27. $10\sqrt{5}[\cos (63°26' + k \cdot 360°) + i \sin (63°26' + k \cdot 360°)]$
29. $-9\sqrt{2} + i9\sqrt{2}$
31. $41i$
33. $-5 - 5i$
35. $5 - i5\sqrt{3}$
37. $-11 - i11\sqrt{3}$
39. $-6.13 + 5.14i$
41. $-8.60 - 12.3i$

Exercises 80, Pages 278–279

1. $4\sqrt{3} + 4i$
3. $-14 + 0i$
5. $2\sqrt{3} + 2i$
7. $3 - i3\sqrt{3}$
9. $-8 + i8\sqrt{3}$
11. $243 + 0i$
13. $0 - 32i$
15. $\frac{1}{2} + \frac{\sqrt{3}}{2}i$
17. $0 + i$
19. $-16\sqrt{3} + 16i$

Exercises 81, Page 282

1. $6(\cos 40° + i \sin 40°)$, $6(\cos 220° + i \sin 220°)$
3. $3(\cos 24° + i \sin 24°)$, $3(\cos 144° + i \sin 144°)$, $3(\cos 264° + i \sin 264°)$
5. $3(\cos 16° + i \sin 16°)$, $3(\cos 106° + i \sin 106°)$, $3(\cos 196° + i \sin 196°)$, $3(\cos 286° + i \sin 286°)$
7. $\cos 30° + i \sin 30°$, $\cos 102° + i \sin 102°$, $\cos 174° + i \sin 174°$, $\cos 246° + i \sin 246°$, $\cos 318° + i \sin 318°$
9. $\sqrt{2}(\cos 30° + i \sin 30°)$, $\sqrt{2}(\cos 210° + i \sin 210°)$
11. $2(\cos 80° + i \sin 80°)$, $2(\cos 200° + i \sin 200°)$, $2(\cos 320° + i \sin 320°$
13. $\sqrt{2}(\cos 75° + i \sin 75°)$, $\sqrt{2}(\cos 165° + i \sin 165°)$, $\sqrt{2}(\cos 255° + i \sin 255°)$, $\sqrt{2}(\cos 345° + i \sin 345°)$
15. $2(\cos 0° + i \sin 0°)$, $2(\cos 72° + i \sin 72°)$, $2(\cos 144° + i \sin 144°)$, $2(\cos 216° + i \sin 216°)$, $2(\cos 288° + i \sin 288°)$
17. $6(\cos 135° + i \sin 135°) = -3\sqrt{2} + i3\sqrt{2}$, $6(\cos 315° + i \sin 315°) = 3\sqrt{2} - i3\sqrt{2}$

19. $3(\cos 0° + i \sin 0°) = 3 + 0i$
$3(\cos 120° + i \sin 120°) = -\frac{3}{2} + \frac{3\sqrt{3}}{2} i$
$3(\cos 240° + i \sin 240°) = -\frac{3}{2} - \frac{3\sqrt{3}}{2} i$

21. $6(\cos 60° + i \sin 60°) = 3 + i3\sqrt{3}$
$6(\cos 180° + i \sin 180°) = -6 + 0i$
$6(\cos 300° + i \sin 300°) = 3 - i3\sqrt{3}$

23. $4(\cos 30° + i \sin 30°) = 2\sqrt{3} + 2i$
$4(\cos 150° + i \sin 150°) = -2\sqrt{3} + 2i$
$4(\cos 270° + i \sin 270°) = 0 - 4i$

25. $3(\cos 45° + i \sin 45°) = \frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2} i$
$3(\cos 135° + i \sin 135°) = -\frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2} i$
$3(\cos 225° + i \sin 225°) = -\frac{3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2} i$
$3(\cos 315° + i \sin 315°) = \frac{3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2} i$

27. $2(\cos 0° + i \sin 0°) = 2 + 0i$
$2(\cos 60° + i \sin 60°) = 1 + i\sqrt{3}$
$2(\cos 120° + i \sin 120°) = -1 + i\sqrt{3}$
$2(\cos 180° + i \sin 180°) = -2 + 0i$
$2(\cos 240° + i \sin 240°) = -1 - i\sqrt{3}$
$2(\cos 300° + i \sin 300°) = 1 - i\sqrt{3}$

29. $\cos 0° + i \sin 0° = 1 + 0i$
$\cos 72° + i \sin 72° = 0.309 + 0.951i$
$\cos 144° + i \sin 144° = -0.809 + 0.588i$
$\cos 216° + i \sin 216° = -0.809 - 0.588i$
$\cos 288° + i \sin 288° = 0.309 - 0.951i$

Exercises 83, Page 294

1. $x < 5$

3. $y > -3$

5. $x > -1$

7. $2 < y < 4$

9. $-9 \leqq s \leqq 2$

11. $v < -4,\ v > \frac{2}{5}$

13. $m < \frac{3 - \sqrt{17}}{2},\ m > \frac{3 + \sqrt{17}}{2}$

15. $\frac{-3 - \sqrt{13}}{4} < w < \frac{-3 + \sqrt{13}}{4}$

17. All real values of y

19. $\frac{2 - 2\sqrt{2}}{3} \leqq z \leqq \frac{2 + 2\sqrt{2}}{3}$

21. $-4 < y < 0,\ 0 < y < \frac{5}{3}$

23. $k < -\frac{11}{3}$

25. $\frac{2}{3} < k < \frac{5}{2}$

27. All real values of k

Exercises 85, Pages 300–301

1. $y < -7, y > 5$
3. $-\frac{7}{2} < x < 3$
5. $u \leqq -\frac{8}{5}, u \geqq 4$
7. $\dfrac{7-\sqrt{65}}{2} < x < \dfrac{7+\sqrt{65}}{2}$
9. $w \leqq \dfrac{-1-\sqrt{261}}{10}, w \geqq \dfrac{-1+\sqrt{261}}{10}$
11. $y < -7, 2 < y < 5$
13. $-2 < m < \frac{5}{2}, m > 4$
15. $-7 < x < 3$
17. $-7 \leqq r \leqq -1, 3 \leqq r \leqq 8$
19. $y < -7, -7 < y < 2, 2 < y < 4$
21. $z < -12$
23. $-5 < v < 11$
25. $y < 2$
27. $-6 < m < -2, m > 2$
29. $-1 < v < \frac{1}{4}, v > 2$
31. $y < -9, y > -7$
33. $x > -6$
35. $2 < x < 6$
37. $x < -5, 1 < x < 4$
39. $y < -7, -3 < y < 8$
41. $y < -4, y > 3$
43. $-\sqrt{6} < y < \sqrt{6}$

Exercises 86, Pages 302–303

1. $m < -12, m > 12$
3. $m < 3, m > 6$
5. $m < -6, m > 6;\ m = \pm 6;\ -6 < m < 6$
7. $0 < k < 3;\ k = 0, 3;\ k < 0, k > 3$
9. $m > \frac{3}{8};\ m = \frac{3}{8};\ m < 0, 0 < m < \frac{3}{8}$
11. $-3 < m < 4;\ m = -3, 4;\ m < -3, m > 4$
13. $m < -12, m > 4;\ m = -12, 0, 4;\ -12 < m < 0, 0 < m < 4$
15. (a) $k < -2, k > 6$; (b) $k = -2, 6$; (c) $-2 < k < 6$
17. (a) $-\frac{7}{2} < k < \frac{7}{2}$; (b) $k = \pm\frac{7}{2}$; (c) $-k < -\frac{7}{2}, k > \frac{7}{2}$

Exercises 87, Pages 307–308

1. $n = 4, a_0 = 7, a_1 = -5, a_2 = 0, a_3 = 15, a_4 = -2;$
$x^4 - \frac{5}{7}x^3 + \frac{15}{7}x - \frac{2}{7} = 0$
3. $n = 7, a_0 = \sqrt{3}, a_1 = 0, a_2 = -4, a_3 = 0, a_4 = 0, a_5 = 0, a_6 = 2\sqrt{3}, a_7 = -7;$
$$y^7 - \frac{4\sqrt{3}}{3}y^5 + 2y - \frac{7\sqrt{3}}{3} = 0$$
5. $v^2 + 3v + 1;\ 0$
7. $x^2 + 6x + 5;\ 15$
9. $z^2 - 6z + 68;\ -384$
11. $3x^3 + 12x^2 + 7x + 15;\ 52$
13. $z^3 + (4+\sqrt{2})z^2 + (4\sqrt{2} - 7)z - 7\sqrt{2};\ 0$
15. $x^3 + 3a^2x - 4a^3;\ -2a^4$
17. $2v^2 + (3+2i)v + 3i;\ 0$
19. $3x^2 + x - 2;\ 0$
21. $3z^2 - z - 4;\ 10$

Exercises 88, Page 311

5. $64;\ -86$
7. $-591;\ -23\frac{7}{8}$
9. $-5 \pm 3i$
11. $\dfrac{2 \pm i\sqrt{11}}{5}$

Exercises 89, Pages 315–316

1. $x^4 - 5x^3 - 7x^2 + 41x - 30 = 0$
3. $x^5 + 2x^4 - 13x^3 + 4x^2 - 30x = 0$
5. $x^4 + 2x^3 - 8x^2 - 24x - 16 = 0$
7. $x^6 + 6x^4 - 32 = 0$
9. $A = 7, B = -4$
11. $A = 2, B = 1, C = 1, D = 3$
13. $\dfrac{-1 \pm i\sqrt{3}}{2}$
15. $-3 \pm i\sqrt{2}$

Exercises 90, Page 319

1. $-1, 2, 3$
3. $3, -3 \pm 2i$
5. $\frac{5}{3}, -4 \pm i\sqrt{2}$
7. $-\frac{1}{4}, \dfrac{3 \pm \sqrt{5}}{2}$
9. $-2, 4, 2 \pm \sqrt{5}$
11. $-\frac{2}{3}, \frac{1}{4}, \pm i$
13. $\frac{3}{2}, \frac{5}{3}, \dfrac{-1 \pm i\sqrt{3}}{2}$
15. $-\frac{3}{4}, \frac{1}{5}, 1, \pm\sqrt{3}$

Exercises 91, Pages 322–323

1. $x^3 + x^2 - 7x + 65 = 0$
3. $x^4 - 2x^3 + 14x^2 - 8x + 40 = 0$
5. $x^3 - (2 + \sqrt{6})x^2 + (5 + 2\sqrt{6})x - 5\sqrt{6} = 0$
7. $x^3 - 9x^2 + 19x + 5 = 0$
9. $x^4 - 6x^3 - 3x^2 + 60x - 70 = 0$
11. $x^4 - 6x^3 + 11x^2 - 2x - 10 = 0$
13. $3 - 2i, \dfrac{-1 \pm \sqrt{10}}{3}$
15. $3 + \sqrt{6}, \dfrac{5 \pm 3\sqrt{3}}{2}$
17. (a) $x^2 - (\sqrt{5} + 3i)x + i3\sqrt{5} = 0$
    (b) $x^3 - \sqrt{5}x^2 + 9x - 9\sqrt{5} = 0$
    (c) $x^4 + 4x^2 - 45 = 0$

Exercises 92, Page 325

1. $x^2 + 5x - 150 = 0$
3. $z^3 + 128z - 448 = 0$
5. $u^4 + u^3 - 2u - 24 = 0$
7. $y^6 + 4y^5 + y^4 + 2y^3 + 17 = 0$
9. $7w^3 - w^2 - 2w + 2 = 0$
11. $4u^3 + 70u^2 - 13{,}000 = 0$
13. $x^3 + 71x^2 + 45x + 68 = 0$
15. $5t^4 - 3t^3 + 7t + 6 = 0$

Exercises 93, Pages 328–329

1.

+	−	i
0	1	2

3.

+	−	i
1	2	0
1	0	2

5.

+	−	i
3	1	0
1	1	2

7.

+	−	i
1	1	2

9.

+	−	i
2	2	2
2	0	4
0	2	4
0	0	6

11.

+	−	i
3	2	2
3	0	4
1	2	4
1	0	6

13.

+	−	i
1	6	2
1	4	4
1	2	6
1	0	8

15.

+	−	i
1	1	$2k - 2$

Exercises 94, Page 331

1. $1 < r_1 < 2$
3. $-4 < r_1 < -3$
5. $-1 < r_1 < 0,\ 0 < r_2 < 1,\ 2 < r_3 < 3$
7. $-5 < r_1 < -4,\ -1 < r_2 < 0,\ 9 < r_3 < 10$
9. $-5 < r_1 < -4,\ 5 < r_2 < 6$

Exercises 95, Pages 334–335

1. 2.16
3. 3.46
5. 0.19
7. -3.424
9. -4.082
11. 6.043
13. -1.784, 2.389, 5.395
15. 0.397, 3.357, $-0.627 \pm 2.521i$
17. 0.445, 2.877
19. 1.883 ft by 3.883 ft by 5.883 ft approx.
21. 3.268 in. approx.

Exercises 96, Page 338

1. $y_1^2 + 10y_1 + 16 = 0$
3. $x_1^3 + 9x_1^2 + 37x_1 + 46 = 0$
5. $2z_1^4 + 40z_1^3 + 299z_1^2 + 990z_1 + 1216 = 0$
7. $v_1^4 - 4v_1^3 + 6v_1^2 - 3v_1 - 2 = 0$
9. $4s_1^3 + 0.6s_1^2 - 0.72s_1 + 19.838 = 0$
11. $x_1^2 + 6x_1 - 1 = 0$
13. $x_1^3 + 6x_1^2 + 27x_1 + 15 = 0$

Exercises 97, Page 344

1. 2.074
3. -0.375
5. 5.063, $-0.532 \pm 1.049i$
7. -4.427, 1.849

Exercises 98, Pages 345–346

1. $x^3 - 2x^2 - 29x + 30 = 0$
3. $x^3 - 8x^2 + 25x - 26 = 0$
5. $x^4 - 4x^3 - 7x^2 + 34x - 24 = 0$
7. $x^4 - 9x^3 + 25x^2 - 21x + 4 = 0$
9. $x^4 - 6x^3 + 12x^2 - 6x - 5 = 0$
11. $b = 80;\ \pm 4,\ 5$
13. $a = 16,\ b = 4;\ 2 \pm \sqrt{6},\ 2 \pm \sqrt{6}$
15. $a = 26;\ 1 - 3\sqrt{3},\ 1,\ 1 + 3\sqrt{3}$
17. $-\dfrac{c}{d}$

Exercises 99, Pages 351–352

1. $\sqrt[3]{2} - \sqrt[3]{4},\ \omega\sqrt[3]{2} - \omega^2\sqrt[3]{4},\ \omega^2\sqrt[3]{2} - \omega\sqrt[3]{4}$
3. $-(\sqrt[3]{4} + 2\sqrt[3]{2}),\ -(\omega\sqrt[3]{4} + \omega^2 2\sqrt[3]{2}),\ -(\omega^2\sqrt[3]{4} + \omega 2\sqrt[3]{2})$
5. $\sqrt[3]{3} - \sqrt[3]{9},\ \omega\sqrt[3]{3} - \omega^2\sqrt[3]{9},\ \omega^2\sqrt[3]{3} - \omega\sqrt[3]{9}$
7. $\dfrac{1}{\sqrt[3]{2}} - \sqrt[3]{2},\ \dfrac{\omega}{\sqrt[3]{2}} - \omega^2\sqrt[3]{2},\ \dfrac{\omega^2}{\sqrt[3]{2}} - \omega\sqrt[3]{2}$
9. $-2 + \sqrt[3]{25} - \sqrt[3]{5},\ -2 + \omega\sqrt[3]{25} - \omega^2\sqrt[3]{5},\ -2 + \omega^2\sqrt[3]{25} - \omega\sqrt[3]{5}$
11. $2\cos 20^\circ,\ 2\cos 140^\circ,\ 2\cos 260^\circ$
13. $k > -3$
15. All values of k

Exercises 100, Page 354

1. 3.411, −2.227, −1.185
3. 4.596, −5.638, 1.042
5. 0.759, −6.064, −3.695
7. 2.532, −0.879, 1.347
9. 4.224, −3.384, −0.840
11. 4.233, −2.656, 1.423

Exercises 101, Page 356

1. $1 \pm \sqrt{3},\ -1 \pm i\sqrt{5}$
3. $5 \pm 2\sqrt{7},\ \pm i\sqrt{3}$
5. $1 \pm i\sqrt{2},\ -2 \pm \sqrt{5}$
7. $-2 \pm \sqrt{10},\ -1 \pm i$
9. $2 \pm \sqrt{2},\ 4 \pm 2\sqrt{3}$
11. $-3 \pm i,\ 1 \pm i\sqrt{3}$

Exercises 103, Pages 364–365

(Each solution is given in the alphabetical order of the unknowns.)

1. (1, 3), (2, 12)
3. $(-4, -7),\ (\frac{5}{2}, -\frac{15}{4})$
5. (2, 4), (4, 2)
7. $(-\frac{5}{2}, -6),\ (3, 5)$
9. $(\frac{11}{4}, \frac{5}{4}),\ (\frac{7}{2}, \frac{5}{2})$
11. $(-3, 1),\ (-3, 1)$
13. $(-\frac{13}{5}, -\frac{21}{5}),\ (2, 5)$
15. $(1, -1),\ (-6, -2)$
17. $(1, \frac{1}{2}),\ (\frac{5}{7}, \frac{11}{14})$
19. $(\frac{3}{2}, 1),\ \left(-\dfrac{3m}{2k}, -\dfrac{k}{m}\right)$

Exercises 104, Pages 367–368

(Each solution is given in the alphabetical order of the unknowns.)

1. $(-5, \pm 3),\ (5, \pm 3)$
3. $(-2, \pm 3),\ (2, \pm 3)$
5. $(-3, \pm 4),\ (3, \pm 4)$
7. $(-3i, \pm 5),\ (3i, \pm 5)$

9. $(-\frac{5}{2}, \pm\frac{5}{2}i)$, $(\frac{5}{2}, \pm\frac{5}{2}i)$

11. $(-3\sqrt{2}, \pm\sqrt{2})$, $(3\sqrt{2}, \pm\sqrt{2})$

13. $\left(-\frac{ab}{\sqrt{a^2-b^2}}, \pm\frac{ab}{\sqrt{a^2-b^2}}\right)$, $\left(\frac{ab}{\sqrt{a^2-b^2}}, \pm\frac{ab}{\sqrt{a^2-b^2}}\right)$, $a \neq b$

Exercises 105, Pages 371–372

(Each solution is given in the alphabetical order of the unknowns.)

1. $(-7, -1)$, $(7, 1)$, $(-1, -7)$, $(1, 7)$
3. $(1, \sqrt{3})$, $(1, -\sqrt{3})$, $(4, 0)$, $(4, 0)$
5. $(-6, 2)$, $(-6, -2)$, $(2, 6)$, $(2, -6)$
7. $(-4, 6)$, $(2, -12)$, $(3, -8)$
9. $(-6, 3)$, $(-4, -2)$, $(2, -1)$, $(3, \frac{3}{2})$
11. $(-1, 2)$, $(1, -2)$, $(-3, -7)$, $(4, 7)$
13. $(-\frac{19}{2}, -\frac{5}{2})$, $(\frac{1}{2}, \frac{5}{2})$
15. $(-2, -5)$, $(2, 5)$, $(-\frac{5}{2}, -4)$, $(\frac{5}{2}, 4)$
17. $(-2, 6)$, $(2, -6)$, $(-1, 1)$, $(1, -1)$
19. $(1, -3)$, $(3, 1)$, $(-2 + i2\sqrt{2}, 4 + i\sqrt{2})$, $(-2 - i2\sqrt{2}, 4 - i\sqrt{2})$
21. $(-3, -3)$, $(3, 3)$, $(-1, -5)$, $(1, 5)$
23. $(-6, 4)$, $(6, -4)$, $(0, -2)$, $(0, 2)$
25. $(-6, -8)$, $(8, 6)$
27. $(4, 9)$, $(9, 4)$
29. $(\frac{1}{2}, 1)$
31. $(16, 243)$, $(81, 32)$

Exercises 106, Pages 374–375

(Each solution is given in the alphabetical order of the unknowns.)

1. $(-2, -1)$, $(2, 1)$, $(-1, -3)$, $(1, 3)$
3. $(-\sqrt{2}, -2\sqrt{2})$, $(\sqrt{2}, 2\sqrt{2})$, $(-2\sqrt{3}, -\sqrt{3})$, $(2\sqrt{3}, \sqrt{3})$
5. $(-1, -4)$, $(1, 4)$, $(-4i, -2i)$, $(4i, 2i)$
7. $(-2, -1)$, $(2, 1)$, $\left(-\sqrt{3}, -\frac{\sqrt{3}}{3}\right)$, $\left(\sqrt{3}, \frac{\sqrt{3}}{3}\right)$
9. $(-2 - \sqrt{7}, -2 + \sqrt{7})$, $(-2 + \sqrt{7}, -2 - \sqrt{7})$, $(2, 3)$, $(3, 2)$
11. $\left(-\frac{2}{3} + \frac{i\sqrt{10}}{3}, -\frac{2}{3} - \frac{i\sqrt{10}}{3}\right)$, $\left(-\frac{2}{3} - \frac{i\sqrt{10}}{3}, -\frac{2}{3} + \frac{i\sqrt{10}}{3}\right)$, $(-2, 5)$, $(5, -2)$

Exercises 107, Pages 375–376

1. 9, 6 or $-6, -9$
3. $\frac{-1 + \sqrt{5}}{2}, \frac{3 - \sqrt{5}}{2}$
5. $(12 - 4\sqrt{3})$ in. by $(12 + 4\sqrt{3})$ in.
7. 240 mph
9. 150 ft by 180 ft
11. 6 in. by 6 in. by 3 in. or $4\frac{10}{11}$ in. by $4\frac{10}{11}$ in. by $5\frac{10}{11}$ in.
13. 300 mph, 400 mph

Exercises 108, Pages 380–381

1. 22 **3.** 9 **5.** 19 **7.** $6ab + cd$

(Each solution is given in the alphabetical order of the unknowns.)

9. $(3, -2)$ **11.** $(6, -8)$ **13.** $(\frac{3}{4}, \frac{1}{2})$ **15.** $(10, 2)$

17. $\left(\dfrac{ac + bd}{a^2 + b^2}, \dfrac{ad - bc}{a^2 + b^2}\right)$ **19.** $(\frac{7}{2}, \frac{5}{2})$ **21.** $(\frac{2}{3}, -\frac{4}{5})$

23. $(\frac{5}{2}, 7)$

Exercises 109, Pages 385–386

1. 4 **3.** -24 **5.** 77 **7.** $b - b^3$

9. $2x - 3y - 13$

(Each solution is given in the alphabetical order of the unknowns.)

11. $(6, -3, 4)$ **13.** $(3, -2, -4)$ **15.** $(-\frac{1}{3}, \frac{7}{4}, \frac{5}{6})$ **17.** $(1, 0, -5)$

19. $\left(\dfrac{c^2 + a^2 - b^2}{2ac}, \dfrac{b^2 + c^2 - a^2}{4bc}, \dfrac{a^2 + b^2 - c^2}{2ab}\right)$ **21.** $(-3, 4, -2)$

Exercises 110, Pages 390–391

1. 5040; 362,880 **3.** (*a*) +; (*b*) −; (*c*) −; (*d*) −; (*e*) +; (*f*) −

11. $\begin{vmatrix} 2 & 3 & -3 \\ 2 & 4 & -3 \\ 1 & -2 & 4 \end{vmatrix}$

Exercises 111, Pages 396–397

1. 0 **3.** -201 **5.** -328 **7.** -36

9. $abcd$ **11.** $1 - ar - bs - ct$ **13.** $x + 6y - z - 9$ **15.** -48

Exercises 112, Pages 400–401

(Each solution is given in the alphabetical order of the unknowns.)

1. $(-2, -1, 2, 3)$ **3.** $(-4, 3, 6, 7)$ **5.** $(1, \frac{1}{2}, \frac{3}{4}, \frac{1}{3})$ **7.** $(-1, -2, 4, 3, 6)$

Exercises 113, Page 405

(Each solution is given in the alphabetical order of the unknowns.)

1. $(11k, 6k, 7k)$ **3.** $(-14k, 17k, 19k)$ **5.** $(7k, k, 5k)$

7. $(40k, 9k, -26k)$ **9.** $(0, 0, 0)$ only **11.** -3; $(4k, -30k, 33k)$

13. 2; $(5k, -6k, 2k)$ and $\frac{7}{5}$; $(5k, -3k, 5k)$ **15.** $(3, -5)$

17. No solution **19.** $(-\frac{1}{4}, \frac{3}{4})$ **21.** $(2 - k, 4k - 2, k)$

23. $\left(\dfrac{2k - 1}{2}, -3k + 2, k\right)$

Exercises 114, Pages 407–408

1. 2520 **3.** 80 **5.** 1320
7. 720 **9.** 1000 **11.** 24

Exercises 115, Pages 411–412

1. 11 **3.** 9 **5.** 11,880
7. 1680; 20,160 **9.** 3,628,800 **11.** 360
13. 64 **15.** 480 **17.** 69,300
19. 302,400 **21.** (3!)[(8!)(7!)(4!)] **23.** 3,386,880

Exercises 116, Pages 415–416

1. (*a*) 20,475; (*b*) 10,660 **3.** 120 **5.** 3003
7. 2002 **9.** 30,888 **11.** 6624
13. 58,212 **15.** 8568; 2380 **17.** 255
19. 57 **21.** 20; $\dfrac{n(n-3)}{2}$ **23.** 210
25. 511 **27.** 113,400

Exercises 117, Pages 421–422

1. $\frac{1}{10}$ **3.** $\frac{1}{5}$ **5.** $\frac{20}{51}$
7. $\frac{1}{9}$ **9.** $\frac{3}{4}$ **11.** $\frac{625}{3888}$
13. $\frac{3997}{4000}$ **15.** $\frac{193}{512}$ **17.** $\frac{286}{20,825}$
19. $\frac{60}{143}$; $\frac{83}{143}$; $\frac{1}{286}$ **21.** $\frac{40}{91}$ **23.** 0.219; 0.781
25. 0.638; 0.995

Exercises 118, Pages 430–431

1. $\dfrac{2}{x-3}+\dfrac{1}{x-1}$ **3.** $\dfrac{-3}{v+8}+\dfrac{4}{v+6}$

5. $5+\dfrac{3}{x-3}+\dfrac{2}{x+2}$ **7.** $\dfrac{2}{w^2}+\dfrac{9}{w-7}$

9. $\dfrac{7}{(z-2)^2}+\dfrac{2}{z}$ **11.** $\dfrac{3}{2t-3}-\dfrac{4}{t+4}$

13. $\dfrac{-5}{2v+1}+\dfrac{3}{3v+2}$ **15.** $\dfrac{-5}{x-1}+\dfrac{31}{x-2}-\dfrac{26}{x-3}$

17. $\dfrac{3}{y-1}+\dfrac{6}{y+4}-\dfrac{2}{y+1}$ **19.** $\dfrac{-1}{(z-3)^2}+\dfrac{2}{z-3}+\dfrac{3}{z+2}$

21. $\dfrac{6}{(w-5)^2}+\dfrac{\frac{1}{2}}{w-5}-\dfrac{\frac{1}{2}}{w+3}$ **23.** $\dfrac{1}{x^2}+\dfrac{1}{x}+\dfrac{1}{(x-2)^2}-\dfrac{1}{x-2}$

25. $\dfrac{2}{(v-3)^2}+\dfrac{1}{v-3}-\dfrac{3}{(v+1)^2}-\dfrac{1}{v+1}$

Exercises 119, Pages 433–434

1. $\dfrac{2}{x} - \dfrac{2x}{x^2 + 3}$

3. $\dfrac{4y - 2}{y^2 + 2y + 5} + \dfrac{5}{y - 2}$

5. $1 + \dfrac{2}{u} + \dfrac{3u - 7}{u^2 + 7}$

7. $\dfrac{4}{2w - 1} - \dfrac{8w + 4}{4w^2 + 1}$

9. $\dfrac{\frac{1}{4}x}{2x^2 + x + 4} + \dfrac{\frac{1}{4}}{2x - 1}$

11. $\dfrac{-\frac{4}{3}}{r + 2} + \dfrac{\frac{1}{3}}{r - 1} + \dfrac{2r - 1}{r^2 + 3r + 1}$

13. $\dfrac{-2}{y + 1} + \dfrac{2}{y - 1} + \dfrac{1}{y^2 + 2}$

15. $\dfrac{x - 2}{x^2 + 1} + \dfrac{2x + 3}{x^2 + 9}$

17. $\dfrac{-v + 1}{v^2 - v + 1} + \dfrac{3v + 3}{v^2 + v + 1}$

19. $\dfrac{5}{(m - 2)^2} + \dfrac{2m + 3}{m^2 + 5}$

21. $\dfrac{-11x}{(x^2 + x + 7)^2} + \dfrac{2x - 3}{x^2 + x + 7}$

23. $\dfrac{1}{(s^2 + 4)^2} - \dfrac{\frac{3}{2}s}{s^2 + 4} + \dfrac{\frac{3}{2}}{s - 2}$

Exercises 120, Pages 444–445

1. 74, 132
3. 73, 224
5. $y = x^3 + 2x - 10$
7. $y = 3x^3 - 6x + 7$
9. $y = x^4 - 5x + 7$
11. $y = \frac{1}{4}x^4 - \frac{1}{2}x^3 + 13$
13. $6n(n + 2)$; $n(n + 1)(2n + 7)$
15. $n(n + 3)$; $\frac{1}{3}n(n + 1)(n + 5)$
21. $\frac{1}{4}n(n + 1)(n + 2)(n + 3)$
23. 1.57321
25. 1.22392 (1.22393 is correct to five decimal places)
27. 1.56798

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